MATH 131
Final Exam
December 16, 1993

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max = 2:15

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1:29 1:20 1:20 1:20 1:45 1:15
6 ? 2:00 last @ 2:15

Name ______________________

162
$\bar{y}$ 115.2
71.1%

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

6 ea

1. Find the following:

a. $\lim\limits_{x \to 5} \dfrac{x^2 - 25}{x(x-5)}$ $= \lim\limits_{x \to 5} \dfrac{\cancel{(x-5)}(x+5)}{x\cancel{(x-5)}} = \dfrac{10}{5} = 2$ all but 1

b. $\lim\limits_{x \to \infty} \dfrac{x^2 - 25}{2x(x-5)}$ $= \dfrac{x^2-25}{2x^2-10x} = \dfrac{1-\frac{25}{x^2}}{2-\frac{10}{x}} \to \frac{1}{2}$ 11

c. $\lim\limits_{x \to 5^+} \dfrac{x^2 - 55}{2x(x-5)}$ $\dfrac{25-55}{10(+)}$ $\dfrac{-}{++}$ $-\infty$ 13

d. $y = \dfrac{x - \cos x}{x^2 - 4}$, $\dfrac{dy}{dx} =$ $\dfrac{(x^2-4)(1+\sin x) - (x-\cos x)(2x)}{(x^2-4)^2}$

all but 2 (those were close)

11

e. $f(x) = \sqrt{x^2 - 3}$, $f''(x) =$

$f'(x) = \frac{1}{2}(x^2-3)^{-1/2}(2x) = \dfrac{x}{\sqrt{x^2-3}}$

$f''(x) = \dfrac{\sqrt{x^2-3} \cdot 1 - x\,\frac{1}{2}(x^2-3)^{-1/2}(2x)}{(x^2-3)}$

$x(x^2-3)^{-1/2}$ or

$x\left(-\frac{1}{2}\right)(x^2-3)^{-3/2}(2x) + (x^2-3)^{-1/2}$

more ops

f. $y = \sec(x^2 - 2)$, $\dfrac{dy}{dx} =$

$\sec(x^2-2)\tan(x^2-2)(2x)$ 14

g. $\dfrac{d}{dx}\left(e^{x^2} + \ln x\right) = e^{x^2} \cdot 2x + \dfrac{1}{x}$ 5

(must chain rule)

42

2. Find the equation of the straight line $3x^2 - 2xy + xy^2 = 2$ at the point (1,1).

$$6x - 2x\frac{dy}{dx} - 2y + x2y\frac{dy}{dx} + y^2 = 0$$

$$6 - 2y' - 2 + 2y' + 1 = 0$$

5

3. What is $\lim_{n \to \infty} \sum_{k=1}^{n} \left(c_k^2 + \sqrt{c_k}\right) \Delta x$ for [2,4] ?

$$\int_2^4 x^2 + \sqrt{x}\, dx$$

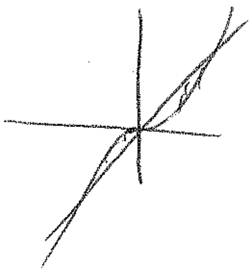
Some

many integrated

15

10

4. Find the area of the region bounded by $y = x^3$ and $y = 4x$. [Hint: sketch the region.]



$x^3 = 4x$

$x^3 - 4x = 0$

$x(x^2 - 4)$

$$\int_0^{2} 4x - x^3\,dx + \int_{-2}^{0} x^3 - 4x\,dx$$

$$= 4\frac{x^2}{2} - \frac{x^4}{4}\Big|_0^{2} + \frac{x^4}{4} - \frac{4x^2}{2}\Big|_{-2}^{0}$$

$= \cancel{32 - 6}$

$= 8 - 4 + 0 - (4 - 8) = 8$

17

15

5. Sketch the graph of $f(x) = x^4 + 12x^3$. Choose ranges to display a complete graph. Find exactly (if any):

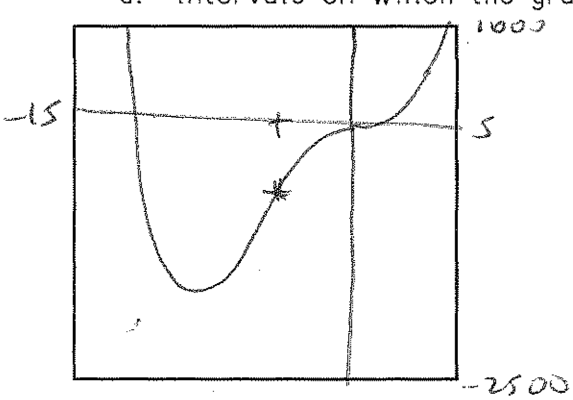
a. x-y coordinates of relative maxima and minina.

b. x-y coordinates of points of inflection.

c. intervals on which the function is increasing, decreasing.

d. intervals on which the graph is concave up, concave down.

11



$y' = 4x^3 + 36x^2 = 0$

$4x^2[x + 9] = 0$

$x = 0,\ x = -9$

$(-9, -2187)$ MIN

NO MAX

$y'' = 12x^2 + 72x$

$12x(x + 6)$

$(0,0), (-6, \cancel{3888})$ -1296

(c) INC $(-9, \infty)$

DEC $(-\infty, -9)$

(d) CC↑ $(-\infty, -6)$ $(0, \infty)$

CC↓ $(-6, 0)$

30

6. The acceleration due to gravity on planet X is 5.6 m/sec/sec (downwards). If an object is tossed upward from 10 m above the surface with a velocity of 28 m/sec:
   a. Find formulas for velocity and height after t seconds.
   b. What is velocity when the object hits the surface? ← too hard

use map

$a = -5.6$

$v = -5.6t + C$

$v = -5.6t + 28$ $\qquad v=0 \quad t=5$

$h = -2.8t^2 + 28t + C$

$h = -2.8t^2 + 28t + 10$ m

$h = 0$ $\quad \cancel{2.8t^2 + 28t =}$

$-2.8t^2 + 28t + 10 = 0$

$t = \dfrac{-28 \pm \sqrt{(28)^2 - 28}}{-5.6}$

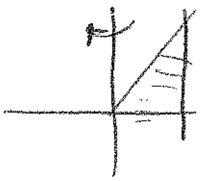
$= \dfrac{-28 \pm 29.933}{-5.6}$

$= 10.34$

$v = -29.93$

6

10 7. A solid is formed by revolving about the y-axis the region bounded by y = 2x, x = 1, and the x-axis. What is the volume?



(a) $\int_0^1 2\pi x\,(2x)\,dx = \int_0^1 4\pi x^2\,dx = \dfrac{4\pi x^3}{3}\Big|_0^1 = \dfrac{4\pi}{3}$

(b) $\int_0^2 \pi - \pi\left(\frac{y}{2}\right)^2 dy = \int_0^2 \pi - \frac{\pi}{4}y^2\,dy = \pi y - \dfrac{\pi y^3}{12}\Big|_0^2$

$= 2\pi - \frac{8\pi}{12} - 0 = 2\pi - \frac{2}{3}\pi = \frac{4}{3}\pi$

10 8. There is 100 pounds of a radioactive byproduct from a factory. It decays at a rate proportional to the amount of radioactive material present, hence $y = 100e^{kt}$ where t is time in years. The half life is 500 years (half will remain after 500 years.)
   a. Show that $k = \dfrac{1}{500} \ln\left(\dfrac{1}{2}\right)$

   b. When will the radioactive material be 10% of the original amount? 11 close

(a) $y = 100e^{kt}$

$50 = 100e^{k500}$

$\frac{1}{2} = e^{k500}$

$\ln\frac{1}{2} = k500$

$k = \dfrac{\ln(.5)}{500}$

(b) $10 = 100e^{\frac{\ln(.5)}{500}t}$

$.1 = e^{\frac{\ln(.5)}{500}t}$

$\ln(.1) = \dfrac{\ln(.5)}{500}t$

$t = \dfrac{500\ln(.1)}{\ln(.5)} =$

$= 1661$

35

9. A 10,000 pound anchor is at the end of 75 feet of iron chain, which weighs 50 pounds per foot. The anchor is raised by winding the chain on a winch on deck.

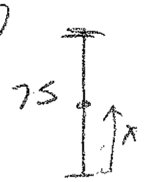
a. How much work is done in raising just the anchor?

b. How much work is done in raising the anchor and chain?

4

ⓐ $75 \cdot \cancel{50}\ 10000 = \cancel{3750}$ ft-lbs   750,000

ⓑ



$\int_0^{75} 50(75-x)\,dx = 50\left(75x - \frac{x^2}{2}\right)\Big|_0^{75} = 50\left(75^2 - \frac{75^2}{2}\right) - 0$

$= 50(\quad) = 140,625$

~~215,625 ft-lbs~~   890,625 ft lbs

10. Integrate the folowing:

5   a. $\int \sqrt{x} - \sin x \, dx = \int x^{1/2} - \sin x \, dx = \frac{x^{3/2}}{3/2} + \cos x + C$

all but 1

most close 3 all

8   b. $\int_0^{\pi/2} e^{2x} + \cos 2x \, dx = \frac{e^{2x}}{2} + \frac{\sin 2x}{2}\Big|_0^{\pi/2}$

most missed

$= \frac{e^{\pi}}{2} + \frac{\sin \pi}{2} - \left(\frac{e^0}{2} + \frac{\sin 0}{2}\right) = \frac{e^{\pi} - 1}{2}$

11.07

7   c. $\int \sqrt{3x - 7}\, dx = \int \sqrt{u}\ \frac{1}{3}du = \frac{1}{3}\frac{u^{3/2}}{3/2} + C$

$u = 3x - 7$

$du = 3dx$

$\frac{1}{3}du = dx$

$= \frac{2}{9}(3x-7)^{3/2} + C$

9

4   d. $\int_3^4 \frac{x}{x^2 - 4}\, dx = \frac{1}{2}\int_5^{12} \frac{1}{u}\, du = \frac{1}{2}\ln u \Big|_5^{12}$

$u = x^2 - 4$ $x = 3\ u = 5$

$du = 2x\,dx$ $x = 4\ u = 12$

$\frac{1}{2}du = x\,dx$

$= \frac{1}{2}(\ln 12 - \ln 5)$

$= .\cancel{240}$

.4378

8

35

11. Compute the derivative of $f(x) = \sqrt{x}$

a. Using formulas: $\frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

b. Using only the definition:

$$\frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h}\,\frac{(\sqrt{x+h}+\sqrt{x})}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{x+h-\cancel{x}}{h(\sqrt{x+h}+\sqrt{x})} = \frac{h}{h(\ )} = \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$\longrightarrow \frac{1}{2\sqrt{x}} \quad \text{as } h \to 0$$

14

10

12. Compute $\int_0^4 3x^3\,dx$

a. using antiderivatives and the Fundamental Theorem (i.e. the easy way)

$$\frac{3x^4}{4}\Big|_0^4 = \frac{3}{4}\,\frac{4^4}{4} - 0 = 3\cdot\cancel{64} = \cancel{48}\,192$$

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b. using the <u>definition</u> (by forming RR_n and then taking the limit as $n \to \infty$).

0 —— 4 $\quad n \quad \Delta x = \frac{4}{n}$

$0 \quad \frac{4}{n} \quad 2\left(\frac{4}{n}\right) \quad 3\left(\frac{4}{n}\right) \cdots n\left(\frac{4}{n}\right)$

$$\sum_{k=1}^{n} 3\left(k\left(\frac{4}{n}\right)\right)^3 \frac{4}{n} = \frac{3\cdot 4^4}{n^4}\sum_{k=1}^{n} k^3$$

$$= \frac{3\cdot 4^4}{n^4}\left(\frac{n(n+1)}{2}\right)^2 = \frac{3\cdot 4^4}{4}\,\frac{\cancel{n^2}(n+1)^2}{n^{\cancel{4}2}}$$

$$= 3\cdot 4^3\,\frac{n+1}{n}\cdot\frac{n+1}{n} = \cancel{3}192\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)$$

$$\longrightarrow 192 \qquad \downarrow 1 \quad \downarrow 1$$

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