Math 131
Final Exam
Dec. 10, 1971

Name KEY

(10) 1. Define Precisely:

a. "The function f is an increasing function on [a,b]."

iff $\forall x_1, x_2$ in $[a,b]$, $x_1 < x_2$, $f(x_1) < f(x_2)$

b. "The function f is continuous at x = a." iff

1. $f(a)$ is def
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a} f(x) = f(a)$

(10) 2. Using only the definiton, find the derivative of $f(x) = 3x^2 + 2$.

$$f'(x) = \lim_{y\to x} \frac{3y^2+2-(3x^2+2)}{y-x} = \lim_{y\to x} \frac{3y^2-3x^2}{y-x}$$

$$= \lim_{y\to x} \frac{3(y-x)(y+x)}{y-x} = \lim_{y\to x} 3(y+x) = 6x$$

(20) 3. Find the following limits:

a. $\lim_{x\to\infty} \frac{x^3}{x^4+1} = \underline{\ 0\ }$

$\frac{\frac{1}{x}}{1+\frac{1}{x^4}}$

b. $\lim_{x\to 1^+} \frac{x}{x^2-x} = \underline{\ \infty\ }$

$\frac{x}{x(x-1)}$

c. $\lim_{x\to 2} \frac{x^2-4}{x-2} = \underline{\ 4\ }$

$\frac{(x-2)(x+2)}{x-2}$

d. $\lim_{x\to 3} \frac{x^3+3}{x^2+1} = \underline{\ \frac{27+3}{9+1}\ } = 3$

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50) 4. Find the following derivatives:

a. $f''(x)$, where $f(x) = x^5 + (x^2+1)^2$.

$f'(x) = 5x^4 + 2(x^2+1)^1(2x)$

$f''(x) = 20x^3 + 4x(2x) + 4(x^2+1)$

$= 20x^3 + 12x^2 + 4$

b. $\frac{\partial}{\partial x}(x^2y^3 + x) = 2xy^3 + 1$

c. $D\left(\frac{x+1}{x-1}\right) = \frac{(x-1) - (x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$

d. $f'(x)$, where $f(x) = \sqrt{x^3+1}\,(\sqrt{x} - 1)$. $= (x^3+1)^{1/2}(x^{1/2}-1)$

$f'(x) = \frac{1}{2}(x^3+1)^{-1/2}(3x^2)(x^{1/2}-1) + (x^3+1)^{1/2}\frac{1}{2}x^{-1/2}$

$\frac{3}{2}x^2\frac{\sqrt{x}-1}{\sqrt{x^3+1}} + \frac{\sqrt{x^3+1}}{2\sqrt{x}}$

e. $f^{VI}(x)$, where $f(x) = x^4 + x^3 - 6$.

$f^{VI} = 0$

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(10) 5. Show that $\lim_{x \to 2} \frac{x^3(x^2-1)}{1+x} = 8$, using only one limit theorem at a time, and that $\lim_{x \to 2} x = 2$.

$$\frac{\lim x^3(x^2-1)}{\lim 1+x} = \frac{(\lim x^3)(\lim x^2-1)}{\lim 1+x}$$

$$= \frac{(\lim x)^3((\lim x^2)-1)}{\lim 1 + \lim x} = \frac{(\lim x)^3((\lim x)^2-1)}{\lim 1 + \lim x}$$

$$= \frac{2^3(2^2-1)}{1+2} = \frac{8(3)}{3} = 8$$

(15) 6. Prove ONE of the following:

a. $D(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

b. If f is continuous for all x in [a,b], and $f'(x) > 0$ for all x in (a,b), then f is strictly increasing on [a,b].

a. $\frac{f(y)g(y) - f(x)g(x)}{y-x} = \frac{f(y)g(y) - f(x)g(y) + f(x)g(y) - f(x)g(x)}{y-x}$

$$= \frac{f(y)-f(x)}{y-x} g(y) + \frac{g(y)-g(x)}{y-x} f(x)$$

lim $= f'(x) g(x) + g'(x) f(x)$

$g' \Rightarrow g$ cont

b. $a \le x_1 < x_2 \le b$

By MVT on $[x_1, x_2]$

$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0 \quad x_1 < c < x_2 \quad$ so $f'(c) > 0$

so since $x_2 - x_1 > 0$, $f(x_2) - f(x_1) > 0$

so $f(x_1) < f(x_2)$

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(10) 7. On what intervals is the graph of the function $f(x) = \frac{x}{1+x}$ concave up? concave down?

$$f'(x) = \frac{(1+x) - x}{(1+x)^2} = \frac{1}{(1+x)^2} = (1+x)^{-2}$$

$$f''(x) = -2(1+x)^{-3} = \frac{-2}{(1+x)^3}$$

$f''(x) > 0$ for $1+x < 0$, $x < -1$; cc up $(-\infty, -1)$

$f''(x) < 0$, $1+x > 0$, $x > -1$; cc ↓ $(-1, \infty)$

(10) 8. Find the equation of the straight line tangent to the curve $y = x^3 + 2x^2 + x + 1$ at the point $(2, 19)$.

$f(x) = x^3 + 2x^2 + x + 1$

$f'(x) = 3x^2 + 4x + 1$

$f'(2) = 3 \cdot 4 + 4 \cdot 2 + 1$

$= 12 + 8 + 1 = 21$

$y - 19 = 21(x - 2)$

$y = 21x - 42 + 19$

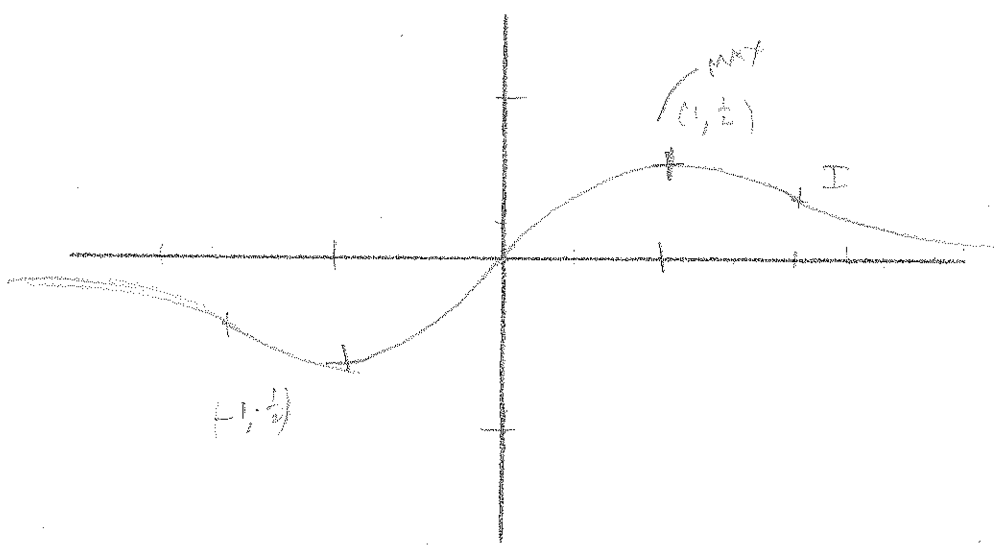
$y = 21x - 23$

(20) 9. Consider $f(x) = \frac{x}{1+x^2}$. Sketch a careful graph of this function using the following information. Plot all critical points and points of inflection, and label them. Indicate the maximum and minimum points.

$f'(x) > 0$ for $-1 < x < 1$
$f'(x) < 0$ for $x > 1$ or $x < -1$
$f'(x) = 0$ for $x = 1, -1$

$f''(x) > 0$ for $x > \sqrt{3}$ or $-\sqrt{3} < x < 0$
$f''(x) < 0$ for $0 < x < \sqrt{3}$ or $x < -\sqrt{3}$
$f''(x) = 0$ for $x = 0, \sqrt{3}, -\sqrt{3}$

cp $(1, \frac{1}{2})$
$(-1, -\frac{1}{2})$
I $(0, 0)$
$(\sqrt{3}, \frac{\sqrt{3}}{4})$
$(-\sqrt{3}, -\frac{\sqrt{3}}{4})$



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(15) 10. A rock is thrown directly upward. Its distance (feet) from the ground at time t (seconds) is given by the function

$$s(t) = 160 + 48t - 16t^2 .$$

a. What is the initial velocity?

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$V(t) = 48 - 32t \qquad V(0) = 48$

b. At what time does it hit the ground?

3

$s(t) = 0 \qquad 160 + 48t - 16t^2 = 0 \qquad t^2 - 3t - 10$

$10 + 3t - t^2 = 0 \qquad (t-5)(t+2) \quad 5$

$t = -2, 5$

c. What is the maximum height that the rock reaches?

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$V(t) = 48 - 32t$

$V = 0 \Rightarrow t = \frac{48}{32} = \frac{3}{2}$

$s(\frac{3}{2}) = 160 + 48 \cdot \frac{3}{2} - 16 \cdot \frac{9}{4}$

$= 160 + 72 - 36$

$= 196$

(15) 11. An astronaut on the moon drops a feather from a height of 4 feet. Assuming the acceleration due to gravity to be 5 feet/sec/sec., downward, find the function $s(t)$ which gives the feather's position at time t.

$a(t) = -5$

$V(t) = -5t + C \qquad V(0) = 0 \Rightarrow C = 0$

$V(t) = -5t$

$s(t) = -\frac{5}{2}t^2 + C \qquad s(0) = 4 \Rightarrow C = 4$

$s(t) = -\frac{5}{2}t^2 + 4$

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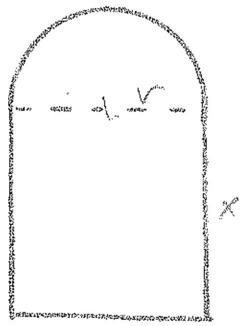
(15) 12. A window is to be made in the shape of a rectangle with a semi-circle on top. If the perimeter is to be 24 feet, what are the dimensions so that the area is a maximum?



$24 = \pi r + 2r + 2x$

$= (\pi + 2) r + 2x \qquad x = 12 - \frac{\pi+2}{2} r$

$A = 2rx + \frac{\pi r^2}{2}$

$A(r) = 2r\left(12 - \frac{\pi+2}{2} r\right) + \frac{\pi r^2}{2}$

$= 24r - (\pi+2)r^2 + \frac{\pi r^2}{2}$

$= 24r + \left(-\frac{\pi}{2} - 2\right) r^2$

$A'(r) = 24 + 2\left(-\frac{\pi}{2} - 2\right) r$

$24 + (-\pi - 4) r = 0$

$A''(r) = -\pi - 4 < 0$ max

$r = \frac{24}{\pi + 4}$

$\pi\left(\frac{24}{\pi+4}\right) +$

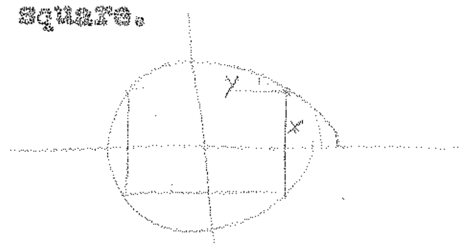
$x = 12 - \frac{\pi+2}{2} \cdot \frac{24}{\pi+4} = 12 - 12\left(\frac{\pi+2}{\pi+4}\right)$

$= 12\left(\frac{\pi+4-(\pi+2)}{\pi+4}\right)$

$= 12\left(\frac{2}{\pi+4}\right)$

$= \frac{24}{\pi+4}$

BONUS Show that the rectangle of maximum area inscribed in a circle is a square.



$x^2 + y^2 = r^2 \qquad r$ fixed

Area $= 4xy$

$A(x) = 4x\sqrt{r^2 - x^2} = 4x(r^2 - x^2)$

$A'(x) = 4(r^2 - x^2)^{1/2} + 4x \cdot \frac{1}{2}(r^2 - x^2)^{-1/2}(-2x)$

$= \cancel{4}(r^2 - x^2)^{1/2} - \cancel{4}x^2(r^2 - x^2)^{-1/2} = 0$

$x^2 = r^2 - x^2$

$2x^2 = r^2$

$x = \frac{r}{\sqrt{2}}$

$y = \frac{r}{\sqrt{2}}$

$x = y$

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