Math 231
Final Exam
Dec. 10, 1973

Name KEY

Show work for full credit!

10 1. Complete the following logarithm table:

x	log x
.5	$-.6931$
1	0
2.	.6931
3	1.0986
4	13862
5	1.6094
6	1.7917
7	1.9459
8	2.0793

$-\ln 2 =$

$.6931 \times 3 = 2.0793$

10 2. If $F(x) = \int_2^x 3t^2 + t\, dt$, then

a. $F(4) = \int_2^4 3t^2 + t\, dt = \frac{3t^3}{3} + \frac{t^2}{2}\Big|_2^4 = 64 + 16 - \frac{8}{2} - \frac{4}{2}$

$64 + \frac{16}{2} - \left(8 + \frac{4}{2}\right) = 62$

$= 80 - 6 = 74$

b. $F'(4) = 3(4)^2 + 4$

$= 48 + 4 = 52$

10 3. Consider $f(x) = 1-x^2$ on the interval $[0,4]$. Let $P = \{\underbrace{0,1}_{1},\underbrace{1,2}_{1},\underbrace{2,2\frac{1}{2}}_{1/2},\underbrace{2\frac{1}{2},3}_{1/2},\underbrace{3,3\frac{1}{4}}_{1/4},\underbrace{3\frac{1}{4},4}_{3/4}\}$ be a partition of $[0,4]$. Write out the following, but do not compute.

a. Lower sum for P

$\underset{0}{(1-1^2)}\cdot 1 + \underset{-3}{(1-2^2)}\cdot 1 + \underset{1-\frac{25}{4} = -\frac{21}{4}}{\left(1-(2\tfrac{1}{2})^2\right)}\left(\frac{1}{2}\right) + \underset{-8}{(1-3^2)}\cdot\frac{1}{2} + \underset{1-\frac{169}{16} = -\frac{153}{16}}{\left(1-(3\tfrac{1}{4})^2\right)}\cdot\frac{1}{4} + \underset{-15}{(1-4^2)}\cdot\frac{3}{4}$

~~$0\cdot 1 + (-3)\cdot 1 +$~~

b. Upper sum for P

$\underset{1}{(1-0^2)}\cdot 1 + \underset{0}{(1-1^2)}\cdot 1 + \underset{-3}{(1-2^2)}\left(\frac{1}{2}\right) + \underset{-\frac{21}{4}}{\left(1-(2\tfrac{1}{2})^2\right)}\cdot\frac{1}{2}$

$+ \underset{-8}{(1-3^2)}\left(\frac{1}{4}\right) + \underset{-\frac{153}{16}}{\left(1-(3\tfrac{1}{4})^2\right)}\cdot\frac{3}{4}$

c. Riemann sum for P using x_i^* as the midpoint of each $[x_{i-1}, x_i]$.

$\underset{3/4}{\left(1-(\tfrac{1}{2})^2\right)}\cdot 1 + \underset{-5/4}{\left(1-(1\tfrac{1}{2})^2\right)}\cdot 1 + \underset{1-\frac{81}{16} = -\frac{65}{16}}{\left(1-(2\tfrac{1}{4})^2\right)}\cdot\frac{1}{2} + \underset{1-\frac{121}{16} = -\frac{105}{16}}{\left(1-(2\tfrac{3}{4})^2\right)}\cdot\frac{1}{2}$

$+ \underset{1-\frac{625}{64} = -\frac{561}{64}}{\left(1-(3\tfrac{1}{8})^2\right)}\cdot\frac{1}{4} + \underset{1-\frac{29^2}{16}}{\left(1-(3\tfrac{5}{8})^2\right)}\cdot\frac{3}{4}$

30

:20

20 4. Find the following derivatives:

a. $D\, xe^{2x}$ $\quad xe^{2x}\cdot 2 + e^{2x} = e^{2x}(2x+1)$

b. $D\sqrt{\sec 2x}$ $\quad \frac{1}{2}(\sec 2x)^{-1/2}(\sec 2x \tan 2x)\cdot 2$

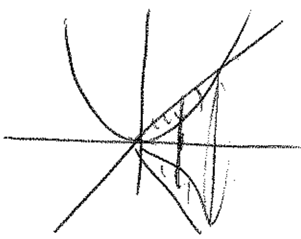
$$= \frac{\sec 2x \tan 2x}{\sqrt{\sec 2x}} = \sqrt{\sec 2x}\ \tan 2x$$

c. $D\, x^2 \ln x$

$$2x\ln x + x^2 \cdot \frac{1}{x} = 2x\ln x + x \quad x(2\ln x + 1)$$

d. $D \sin(e^x+1)$ $\quad = \cos(e^x+1)\cdot(e^x) = e^x\cos(e^x+1)$

10 5. The region bounded by the curves $y = x^2$ and $y = x$ is rotated about the x-axis to form a solid. Find its volume.

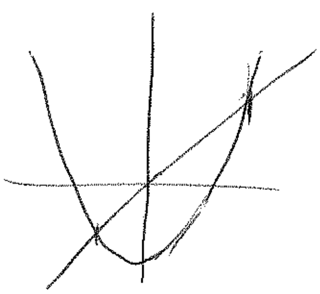


$$A(x) = \pi x^2 - \pi (x^2)^2$$

$$V = \int_0^1 \pi x^2 - \pi x^4\, dx = \frac{\pi x^3}{3} - \frac{\pi x^5}{5}\Big|_0^1$$

$$= \left(\frac{1}{3} - \frac{1}{5}\right)\pi = \frac{2}{15}\pi$$

10 6. Find the area bounded by the curves $y = x$ and $y = x^2-2$.



$$x = x^2 - 2 \qquad x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$\int_{-1}^{2} x - (x^2-2)\, dx = \int_{-1}^{2} x - x^2 + 2\, dx$$

$$= \frac{x^2}{2} - \frac{x^3}{3} + 2x\Big|_{-1}^{2} = \frac{4}{2} - \frac{8}{3} + 4 - \left(\frac{1}{2} - \frac{-1}{3} - 2\right)$$

$$= 6 - \frac{8}{3} - \frac{1}{2} + \frac{1}{3} + 2 = 8 - \frac{1}{2} - \frac{9}{3} = \frac{\cancel{48-3-14}}{6}$$

$$= \cancel{31}\ 4\tfrac{1}{2}$$

70

10 7. Expand $\dfrac{x^2-7x+8}{(x-2)(x-3)^2}$ into a sum of partial fractions.

$$\frac{x^2-7x+8}{(x-2)(x-3)^2} = \frac{A}{x-2} + \frac{B}{x-3} + \frac{C}{(x-3)^2} = \frac{\cancel{1/2}\;-2}{x-2} + \frac{1/2\;\;3}{x-3} - \frac{2\;\;-4}{(x-3)^2}$$

$$x^2-7x+8 = A\overset{x^2-6x+9}{(x-3)^2} + B\overset{x^2-5x+6}{(x-2)(x-3)} + C(x-2)$$

$x=2$ $\quad 4-\overset{-2}{14}+8 = A(-1)^2 + 0 + 0$

$$\boxed{A = \frac{2}{4} = \frac{1}{2}\;\; -2}$$

$x=3$ $\quad 9-21+8 = 0+0+C(1)$

$$\boxed{C = \frac{-4}{2} = -4}$$

$x^2-7x+8 = (A+B)x^2 + (-6A-5B+C)x + 9A+6B-2C$

$A+B=1 \qquad -6A-5B+C=-7 \qquad 9A+6B-2C=8$

$-2+B=1$

$\boxed{B=3}$

ch
$-15 \quad -4$
$+12 \quad +18$
$-3-2+(-2)=-7$

$\frac{18}{2}+\frac{3}{2}+2=8$ $\quad 18+8$

10 8. Radioactive material decomposes at a rate proportional to the quantity present.. Suppose that 100 g. decays to 25 g. after 1000 hours. How long will it take 100 g. to decay to 10 g.?

$f(t) = Ce^{kt}$

$f(0) = 100$

$f(1000) = 25$

Find $f(t) = 10$?

$C = 100$

$25 = 100e^{k(1000)}$

$\frac{1}{4} = e^{k\cdot 1000}$

$k\cdot 1000 = \ln 1/4$

$k = \dfrac{\ln 1/4}{1000}$

$f(t) = 100\, e^{t\frac{\ln 1/4}{1000}}$

$e^{t\frac{\ln 1/4}{1000}} = \dfrac{1}{10}$

$t\,\dfrac{\ln 1/4}{1000} = -\ln 10$

$$\boxed{t = \frac{-\ln 10}{\ln 1/4}\cdot 1000}$$

$= \dfrac{\ln 10}{\ln 4}\cdot 1000$

10 9. $\int x^3 - 5x - \frac{7}{x^2} + \frac{2}{x+1} - x\sqrt{x}\,dx = \frac{x^4}{4} - \frac{5x^2}{2} - 7\frac{x^{-1}}{-1} + 2\ln|x+1| + \frac{x^{5/2}}{5/2} + C$

$\frac{x^4}{4} - \frac{5x^2}{2} + \frac{7}{x} + 2\ln|x+1| + \frac{2}{5}x^{5/2} + C$

10 10. $\int_1^2 e^{2x-1} + x^e + 2^x\,dx = \frac{e^{2x-1}}{2} + \frac{x^{e+1}}{e+1} + \frac{e^{x\ln 2}}{\ln 2}\Big|_1^2$

$= \frac{e^3}{2} + \frac{2^{e+1}}{e+1} + \frac{2^2}{\ln 2} - \left(\frac{e^1}{2} + \frac{1}{e+1} + \frac{2}{\ln 2}\right)$

$= \frac{e^3 - e}{2} + \frac{2^{e+1} - 1}{e+1} + \frac{4-2}{\ln 2}$

10 11. $\frac{1}{2}\int_1^2 2(x+1)(x^2+2x+3)^7\,dx = \frac{1}{2}\,\frac{(x^2+2x+3)^8}{8}\Big|_1^2$

$= \frac{1}{16}(4+4+3)^8 - \frac{1}{16}(1+2+3)^8$

$= \frac{11^8 - 6^8}{16}$

10 12. $-\int_{\pi/6}^{\pi} -(\sin x)e^{1+\cos x}\,dx \quad -e^{1+\cos x}\Big|_{\pi/6}^{\pi}$

$= -e^{1+\cos\pi} + e^{1+\cos\pi/6}$

$= -e^0 + e^{1+\frac{\sqrt{3}}{2}} = e^{1+\frac{\sqrt{3}}{2}} - 1$

10 13. $\frac{1}{6}\int_2^4 \frac{6x}{3x^2+3}\,dx = \frac{1}{6}\ln(3x^2+3)\Big|_2^4$

$= \frac{1}{6}\left(\ln(51) - \ln(15)\right)$

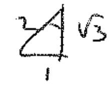
$= \frac{1}{6}\ln\frac{51}{15} = \frac{1}{6}\ln\left(\frac{17}{5}\right)$

50

10 14. $\displaystyle\int \frac{\sec^2 x}{(1+\tan x)^2}\,dx = \int \frac{1}{u^2}\,du = \frac{u^{-1}}{-1} + C = -\frac{1}{(1+\tan x)} + C$

$u = 1 + \tan x$

$du = \sec^2 x\,dx$



10 15. $\displaystyle\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx = \frac{(\sin^{-1} x)^2}{2}\Bigg|_0^{1/2} = \frac{\left(\sin^{-1}(\frac{1}{2})\right)^2}{2} - \frac{(\sin^{-1} 0)^2}{2}$

$u = \sin^{-1} x$

$du = \frac{1}{\sqrt{\ }}\,dx$

$\displaystyle = \frac{(\pi/6)^2}{2} - 0 = \frac{\pi^2}{72}$

$u = 1-x$ $\quad x = 1-u$

$du = -dx$ $\quad x^2 = (1-u)^2$

$\int \frac{(1-u)^2}{u}\,du$

$= \int \frac{1}{u} - 2 + u\,du$

$\ln|u| - 2u + \frac{u^2}{2}$

$\ln|1-x| - 2(1-x) + \frac{(1-x)^2}{2} + C$

10 16. $\displaystyle\int_{-3}^{-2} \frac{x^2}{1-x}\,dx = \int_{-3}^{-2} -x - 1 + \frac{1}{1-x}\,dx$

$$
\begin{array}{r}
-x - 1 \\
1-x\,\overline{)\,x^2\qquad} \\
\underline{+x^2 + x} \\
+x \\
\underline{+x - 1} \\
1
\end{array}
$$

$\displaystyle = -\frac{x^2}{2} - x + \ln|x-1|\Bigg|_{-3}^{-2}$

$\displaystyle = -\frac{4}{2} - (-2) + \ln|-3|$

$\displaystyle - \left(\frac{9}{2} - (-3) + \ln|-4|\right)$

$\displaystyle = +\frac{9}{2} - 3 \mp \ln 4 + \ln 3 = +\frac{3}{2} + \ln 4 \mp \ln 3$

10 17. $\displaystyle\int_1^2 (x+1)\sqrt{2x-1}\,dx = \int_{x=1}^{2} \frac{u+3}{2}\sqrt{u}\,du = \int_1^{3/2} \frac{u^{3/2} + 3u^{1/2}}{2}\,du$

$u = 2x - 1$ $\quad x = 1 \quad u = 1$

$du = 2dx$ $\quad x = 2 \quad u = \frac{3}{2}$

$x = \frac{u+1}{2}$

$x + 1 = \frac{u+3}{2}$

$\displaystyle = \frac{u^{5/2}}{5} + \frac{3u^{3/2}}{3}\Bigg|_1^{3/2}$

$\displaystyle = \frac{(3/2)^{5/2}}{5} + \frac{3(\frac{3}{2})^{3/2}}{3} - \frac{1}{5} - \frac{2}{3}$

$\displaystyle = \frac{(\frac{3}{2})^{5/2} - 1}{5} + \frac{\cancel{2}\,(\frac{3}{2})^{3/2} - 2}{\cancel{3}}$

40

10 18. $\int x \cos x \, dx = x \sin x - \int \sin x \, dx$

$u = x \quad dv = \cos x \, dx$

$du = dx \quad v = \sin x$

$= x \sin x + \cos x + C$

10 19. $\int \frac{x^4+1}{x^3-1} \, dx = \int x + \frac{x+1}{x^3-1} \, dx = \int x + \frac{2/3}{x-1} + \frac{-\frac{2}{3}x - \frac{1}{3}}{x^2+x+1} \, d$

$$\begin{array}{r} x \\ x^3-1 \,\big)\, \overline{x^4+1} \\ \underline{x^4-x} \\ x+1 \end{array}$$

$= \frac{x^2}{2} + \frac{2}{3} \ln|x-1| - \frac{1}{3} \ln(x^2+x+1) + C$

$\frac{x+1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$

$x+1 = A(x^2+x+1) + (Bx+c)(x-1)$

$Bx^2 + Cx - Bx - C$

$x = 1$

$2 = 3A + 0$

$A = \frac{2}{3}$

$x+1 = (A+B)x^2 + (A+C-B)x + A - C$

$A+B = 0 \qquad B = -\frac{2}{3}$

~~ch~~

$A+C-B = 1$

$\frac{4}{3} + C = 1$

$C = 1 - \frac{4}{3} = -\frac{1}{3}$

ch $A - C = 1$ ✓

20

Extra Credit

A. Derive the integration by parts formula.

B. Describe the process by which the integral is defined, including when a function is integrable.