Math 231
Final Exam
June 4, 1974

Name KEY

3 ea I. Short answer, no partial credit.

a. $\ln x = \int_1^x \frac{1}{t}\,dt \;, x > 0$

b. $y = e^x$ if and only if $x = \ln y$

c. $\int 3e^{-4x}\,dx = \frac{3e^{-4x}}{-4} + c$

d. $D\,e^3 = 0$

e. $\int \ln 3\,dx = (\ln 3)x + C$

f. For what values of x is ln x negative? $0 < x < 1$

g. $\cos^{-1}(-1/2) = \frac{2\pi}{3}$

h. $\sin \frac{3\pi}{4} = \frac{1}{\sqrt{2}}$

i. $D\sqrt{e^x} = \frac{1}{2}e^{x/2}$

j. $\int x^3 - 6x + 2\,dx = \frac{X^4}{4} - \frac{6x^2}{2} + 2x + c$

k. $\int \cos 2x\,dx = \frac{\sin 2x}{2} + c$

l. $\int \frac{3}{2x-7}\,dx = \frac{3}{2}\ln|2x-7| + C$

m. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$

n. $\sin^{-1}(\sin \frac{11\pi}{4}) = \frac{\pi}{4}$

o. $e^{\ln x} = X$

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I cont.

p. $D \tan x =$ $\sec^2 x$

q. $\int 2x\sqrt{x}\, dx =$ ~~$3x^{1/2}$~~ $\int 2x^{3/2}dx = \frac{4}{5}x^{5/2} + c$

r. $\log_2 \sqrt{32} =$ $\frac{1}{2}\ln_2 32 = \frac{1}{2}\log_2 16\cdot 2 = \frac{1}{2}\cdot 5 = \frac{5}{2}$

(20) II. Compute the following derivatives:

a. $D\, xe^{x^2+1} =$ $x e^{x^2+1}(2x) + e^{x^2+1}$

$= e^{x^2+1}(2x^2+1)$

b. $D \tan^{-1}(x+3) =$ $\frac{1}{(x+3)^2+1}$

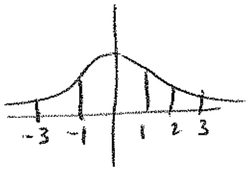
c. $D(\sin x \tan 2x) =$ $\cos x \tan 2x + \sin x \; 2\sec^2 2x$

$\cos x \tan 2x + 2\sin x \sec^2 2x$

d. $D(1+\cos 2x)^{3/2} =$ $\frac{3}{2}(1+\cos 2x)^{1/2}(-\sin 2x\cdot 2)$

$= -3\sqrt{1+\cos 2x}\,\sin 2x$

(10) III. Write out the upper and lower sums for $f(x) = \frac{1}{1+x^2}$ on $[-3,3]$ and partition $P = \{-3,-1,1,2,3\}$.



$[-3,-1]$ $\Delta x = 2$ $M = f(-1) = \frac{1}{2}$ $m = f(-3) = \frac{1}{10}$

$[-1,1]$ $\Delta x = 2$ $M = f(0) = 1$ $m = f(-1 \text{ or } 1) = \frac{1}{2}$

$[1,2]$ $\Delta x = 1$ $M = f(1) = \frac{1}{2}$ $m = f(2) = \frac{1}{5}$

$[2,3]$ $\Delta x = 1$ $M = f(2) = \frac{1}{5}$ $m = f(3) = \frac{1}{10}$

$S(P) = \frac{1}{10}\cdot\overset{2}{\cancel{\frac{1}{2}}} + \frac{1}{2}\cdot 2 + \frac{1}{5}\cdot 1 + \frac{1}{10}\cdot 1$

$T(P) = \frac{1}{2}\cdot 2 + 1\cdot 2 + \frac{1}{2}\cdot 1 + \frac{1}{5}\cdot 1$

(10) IV. Write out the partial fractions expansion of each of the following. DO not solve!

a. $\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$

b. $\frac{2}{(x-1)^2(x+2)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2} + \frac{E}{(x+2)^3}$

c. $\frac{x}{(x^2+x+1)^2(x-5)} = \frac{A}{x-5} + \frac{Bx+C}{x^2+x+1} + \frac{Dx+E}{(x^2+x+1)^2}$

10ea V. Compute the following integrals:

a. $\int_{-1}^{1} \frac{x^2}{x^2+1}\,dx = \int_{-1}^{1} 1 - \frac{1}{x^2+1}\,dx = x - \tan^{-1}x \Big|_{-1}^{1}$

$= 1 - \tan^{-1}1 - (-1 - \tan^{-1}(-1))$

$= 2 - \frac{\pi}{4} + -\frac{\pi}{4}$

$= 2 - \frac{\pi}{2}$

① $x^2+1 \overline{) x^2}$ quotient 1, $\frac{x^2+1}{-1}$

~~$\int_{-2}^{2} 1 - \frac{1}{x^2+1}\,dx = x - \tan^{-1}x \Big|_{-2}^{2}$~~

~~$= 2 - \tan^{-1}2 \overset{-(-2)}{+} \tan^{-1}(-2)$~~

~~$= 4 - \tan^{-1}2 + \tan^{-1}(-2)$~~

$\tan u = x$

$\sec^2 u\,du = dx$

$\frac{1}{\sqrt{1+x^2}} = \cos u$

$\int_{x=1}^{x=1} \frac{\tan^2 u}{\sec^2 u} \sec^2 u\,du$

$x=-1 \quad u = -\pi/4$

$x = 1 \quad u = \pi/4$

$= \int_{x=-1}^{1} \sec^2 u - 1\,du = \tan u - u \Big|_{x=-1\ \ -\pi/4}^{\pi/4} = 1-(-1) - \frac{\pi}{4} - (-\frac{\pi}{4})$

$= 2 - \frac{\pi}{2}$

b. $\int x^2 \cos x\,dx = x^2 \sin x - \int 2x \sin x\,dx = x^2 \sin x - 2\int x \sin x\,dx$

$u = x^2 \quad dv = \cos x\,dx$

$du = 2x\,dx \quad v = \sin x\,dx$

$u = x \quad dv = \sin x$

$du = dx \quad v = -\cos x\,dx$

$\int x \sin x\,dx = -x\cos x - \int -\cos x\,dx$

$= -x\cos x + \int \cos x\,dx$

$= -x\cos x + \sin x + C$

$= x^2 \sin x \overset{+}{-} 2x\cos x \overset{-}{+} 2\sin x + C$

~~$u = x \quad dv = \frac{x}{x^2+1}$~~

~~$du = dx \quad v = \ln(x^2+1)$~~

~~$u = x^2 \quad dv = \frac{1}{x^2+1}$~~

~~$du = 2x \quad v = \tan^{-1}x$~~

~~$u = x^2+1 \quad \frac{1}{2}\int \frac{\sqrt{u-1}}{u}\,du$~~

~~$du = 2x\,dx$~~

~~$x = \sqrt{u-1}$~~

V. cont.

c. $\int \frac{\sin^3 x}{\cos x}\, dx =$ $\int \frac{\sin^2 x}{\cos x}\, dx = \int \frac{1-\cos^2 x}{\cos x} \sin x\, dx$

$u = \cos x$
$du = -\sin x\, dx$ $= -\int \frac{1-u^2}{u}\, du = -\int \frac{1}{u} - u\, du$

$= \int -\frac{1}{u} + u\, du = -\ln|u| + \frac{u^2}{2} + c$

$= -\ln|\cos x| + \frac{\cos^2 x}{2} + c$

d. $\int \frac{5x^2+12x+1}{(x-1)(x+2)^2}\, dx =$

$$\frac{5x^2+12x+1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$$

$$5x^2+12x+1 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$$

$x=1$ $18 = 9A$, $A = 2$ $x=-2$ $-3 = -3C$, $C = 1$ $x=-1$ $-6 = 2(1) + 2B - 2$, $B = 3$

ck $x=2$ ✓
$45 = 32 + 12 + 1$

$$= \int \frac{2}{x-1}\, dx + \int \frac{3}{x+2}\, dx + \int \frac{1}{(x+2)^2}\, dx$$

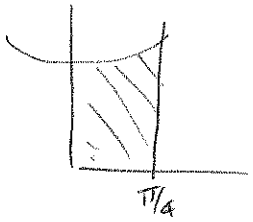
$$= 2\ln|x-1| + 3\ln|x+2| - \frac{1}{x+2} + C$$

(15)

VI. a. The region bounded by the curves y=sec x, x = $\frac{\pi}{4}$, and the coordinate axes is rotated about the x axis to form a solid. What is its volume?

b. Find the area of the region bounded by the curves $y = x^2-2x$ and $y = x$.

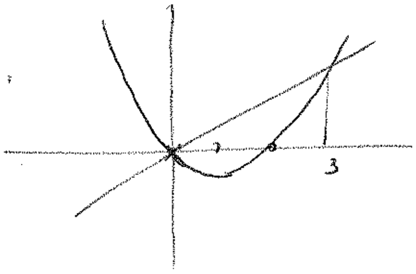
a.



$$V = \int_0^{\pi/4} \pi \sec^2 x \, dx = \pi \tan x \Big|_0^{\pi/4}$$

$$= \pi \tan \pi/4 - 0 = \pi$$

b.



$x(x-2)$

$x^2-2x=x$

$x^2-3x=0$

$x(x-3)=0$

$x=0 \quad x=3$

$$\int_0^3 x-(x^2-2x)\,dx = \int_0^3 -x^2+3x\,dx = -\frac{x^3}{3}+\frac{3x^2}{2}\Big|_0^3$$

$$= -9+\frac{27}{2}-0 = \frac{9}{2}$$

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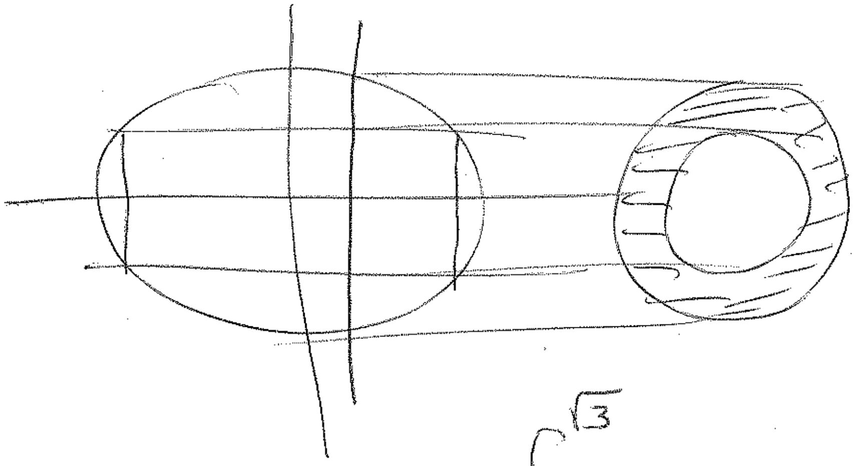
Extra Credit!

1. Derive the integration by parts formula.

2. A 1 inch hole is drilled through the center of a sphere with radius 2 inches. What is the volume of what is left?

2.



$y = \sqrt{4 - x^2}$

$y = 1 = \sqrt{4 - x^2}$

$4 - x^2 = 1$

$x^2 = 3$

$x = \pm\sqrt{3}$

$$\int_{-\sqrt{3}}^{\sqrt{3}} \pi(4 - x^2) - \pi$$

$$\int_{-\sqrt{3}}^{\sqrt{3}} 3\pi - \pi x^2 \, dx$$

$$3\pi x - \frac{\pi x^3}{3} \Big|_{-\sqrt{3}}^{\sqrt{3}}$$

$$= 3\pi\sqrt{3} - \frac{\pi 3\sqrt{3}}{3} - \left(-3\pi\sqrt{3} + \frac{\pi 3\sqrt{3}}{3}\right)$$

$$= 6\sqrt{3}\pi - 2\pi\sqrt{3} = 4\pi\sqrt{3}$$