Math 231
Test I
April 12, 1974

Name *KEY*

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Show work for partial credit!

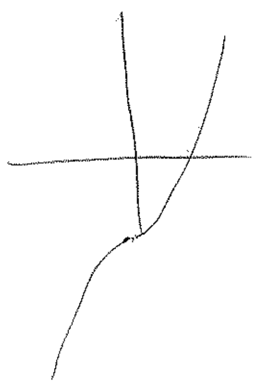
(5) 1. State precisely the Fundamental Theorem of Calculus:

If $f(x)$ is cont for all x in $[a,b]$, and $F'(x) = f(x)$ $\forall x$ in $[a,b]$
~~then~~

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

(15) 2. Do NOT carry out arithmetic:

a. Write out the upper sum for the function $f(x) = x^3 - 3$ on the interval $[0,5]$ using the partition $P = \{0, 2, 3, 4, 4\frac{1}{2}, 5\}$.



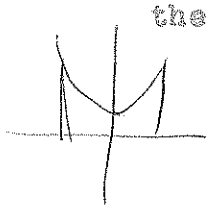
$M_i = f(x_i)$

$$T(P) = f(2)\cdot 2 + f(3)\cdot 1 + f(4)\cdot 1 + f(4\tfrac{1}{2})\cdot\tfrac{1}{2} + f(5)\cdot\tfrac{1}{2}$$

$$= (2^3-3)\cdot 2 + (3^3-3)\cdot 1 + (4^3-3)\cdot 1 + ((4\tfrac{1}{2})^3-3)\cdot\tfrac{1}{2} + (5^3-3)\cdot\tfrac{1}{2}$$

$$5\cdot 2 + 24\cdot 1 + 61\cdot 1 + \quad 122\cdot\tfrac{1}{2}$$

b. Write out the lower sum for $f(x) = x^2+1$ for x in $[-3,3]$ with the partition $P = \{-3, -1, 1, 3\}$.



$m_1 = f(-1)$
$m_2 = f(0)$
$m_3 = f(1)$

$$T(P) = ((-1)^2+1)\cdot 2 + (0^2+1)\cdot 2 + (1^2+1)\cdot 2$$

$$= 2\cdot 2 + 1\cdot 2 + 2\cdot 2$$

$$= 10$$

(10) 3. Write out:

a. $$\sum_{i=2}^{5} \frac{i+3}{3} = \frac{2+3}{3} + \frac{3+3}{3} + \frac{4+3}{3} + \frac{5+3}{3}$$

b. $$\sum_{k=3}^{7} \frac{k\,3^{k+1}}{2} = \frac{3\cdot 3^4}{2} + \frac{4\cdot 3^5}{2} + \frac{5\cdot 3^6}{2} + \frac{6\cdot 3^7}{2} + \frac{7\cdot 3^8}{2}$$

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(20) 4. Consider $f(x) = 1-2x^2$ on $[0,1]$. Compute the upper sum $T(P_n)$ where P_n is the regular partition of $[0,1]$ with n subintervals of equal length, and find $\lim_{n\to\infty} T(P_n)$. Find the integral a shorter way.

$$\Delta x_i = \frac{1}{n} \qquad P_n = \left\{ \frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n} = 1 \right\}$$

$f' = -4x < 0$, $x > 0$ dec

sth $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ $\quad m_i = f\left(\frac{i}{n}\right) = 1 - 2\left(\frac{i}{n}\right)^2$

$M_i = f\left(\frac{i-1}{n}\right) = 1 - 2\left(\frac{i-1}{n}\right)^2$

$$T(P_n) = \sum_{i=1}^{n} \left(1 - 2\left(\frac{i-1}{n}\right)^2\right) \cdot \frac{1}{n}$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left(1 - 2\frac{(i-1)^2}{n^2}\right)$$

$$= \frac{1}{n} \sum_{i=1}^{n} 1 - \frac{1}{n} \sum_{i=1}^{n} 2\frac{(i-1)^2}{n^2}$$

$$= 1 - \frac{2}{n^3} \sum_{i=1}^{n} (i-1)^2$$

or

$$1 - \frac{2}{n^3} \sum (i^2 - 2i + 1)$$

$$= 1 - \frac{2}{n^3} \left[\sum i^2 - 2\sum i + n\right]$$

$$= 1 - \frac{2}{n^3} \left(\frac{n(n+1)(2n+1)}{6} - 2\frac{n(n+1)}{2} + n\right]$$

$$= 1-$$

$$= 1 - \frac{2}{n^3} \sum_{k=1}^{n-1} k^2$$

$$= 1 - \frac{2}{n^3} \frac{(n-1)\overset{n}{(n-1+1)}\overset{2n-1}{(2(n-1)+1)}}{6}$$

$$= 1 - \frac{\cancel{2}}{3} \frac{n-1}{n} \cdot \frac{2n-1}{n}$$

$$\lim_{n\to\infty} T(P_n) = \lim_{n\to\infty} \left(1 - \frac{1}{3}\left(\frac{n-1}{n}\right)\left(\frac{2n-1}{n}\right)\right)$$

$$= \lim_{n\to\infty} \left(1 - \frac{1}{3}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)\right) = 1 - \frac{2}{3}$$

$$= \frac{1}{3}$$

ck $\displaystyle\int_0^1 1 - 2x^2\,dx$

$$= x - \frac{2x^3}{3}\Big|_0^1$$

$$= 1 - \frac{2}{3} - 0 = \frac{1}{3}$$

Tech 5. Compute the following integrals:

a. $\displaystyle\int_0^2 6x^3 - 6x + 7\,dx = \frac{6x^4}{4} - \frac{6x^3}{2} + 7x\Big|_0^2$

$$= \frac{6 \cdot 16}{4} - \frac{6 \cdot 4}{2} + 14 - 0 = 24 - 12 + 14$$

$$= 26$$

(7ea) 5 b. $\int_1^4 3x^2-3\sqrt{x}+\frac{4}{x^2}\,dx = \int_1^4 3x^2-3x^{1/2}+4x^{-2}\,dx$

$$= \frac{3x^3}{3} - \frac{3x^{3/2}}{3/2} + \frac{4x^{-1}}{-1}\Big|_1^4 = 4^3 - 2(4)^{3/2} - 4\cdot\frac{1}{4} - (1-2\cdot 1 + 4)$$

$$= 64 - 16 \cancel{-1} \cancel{-1+2} + 4 = \cancel{44}\ 52$$

c. $\int_4^5 (2x-9)^4\,dx = \frac{1}{2}\int_4^5 (2x-9)^4\cdot 2\,dx = \frac{1}{2}\frac{(2x-9)^5}{5}\Big|_4^5$

B

$$= \frac{1}{10}(1)^5 - \frac{1}{10}(-1)^5 = \frac{1}{10} + \frac{1}{10} = \frac{1}{5}$$

d. $\int_1^5 \frac{3}{\sqrt{2x-1}}\,dx = \frac{3}{2}\int_1^5 (2x-1)^{-1/2}2\,dx$

$$\frac{3}{2}\frac{(2x-1)^{1/2}}{1/2}\Big|_1^5 = 3(10-1)^{1/2} - 3(2-1)^{1/2}$$

$$= 9 - 3 = 6$$

e. $\int (x^3-3)^5x^2\,dx = \frac{1}{3}\int (x^3-3)^5 3x^2\,dx = \frac{1}{3}\frac{(x^3-3)^6}{6} + C$

$$= \frac{(x^3-3)^6}{18} + C$$

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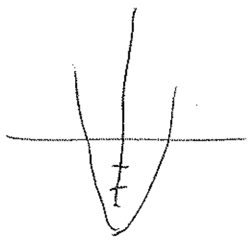
(15) 7 6. Find the area of the regions bounded by the following curves:

a. $y = x^2-4$ and the x-axis.

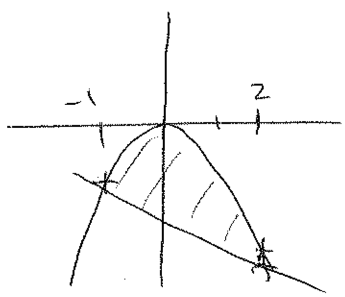


$x = \pm 2$ area $= -\int_{-2}^{2} x^2-4\,dx = -\left[\frac{x^3}{3} - 4x \Big|_{-2}^{2}\right]$

$= -\left(\frac{8}{3}-8\right) + \left(\frac{-8}{3}+8\right) = 16 - \frac{16}{3}$

$= \frac{48-16}{3} = \frac{32}{3}$

8 b. $y = -x^2$ and the straight line which cuts the curve at (-1,-1) and (2,-4).



$m = \frac{-1-(-4)}{-1-2} = \frac{-1+4}{-3} = \frac{3}{-3} = -1$

$\frac{y+1}{x+1} = -1$ $y+1 = -x-1$

$y = -x-2$

area $= \int_{-1}^{2} -x^2-(-x-2)\,dx$

$= \int_{-1}^{2} -x^2 + x + 2\,dx = -\frac{x^3}{3} + \frac{x^2}{2} + 2x \Big|_{-1}^{2}$

$-\frac{8}{3} + \frac{4}{2} + 4$

$-\left(\frac{1}{3} + \frac{1}{2} - 2\right)$

$-\frac{9}{3} + 8 - \frac{1}{2}$

$\frac{-18+48-3}{6} =$

$\frac{27}{6} = \frac{9}{2}$

$= -\frac{\cancel{16}}{\cancel{3}} + \frac{4}{2} + 4 - \left(\frac{1}{3} + \frac{1}{2} + 2\right)$

$= -\cancel{4} + \cancel{2} + \cancel{4} + \frac{1}{4} - \frac{1}{2} - \cancel{2} = \frac{1}{4}$

$\frac{7}{3} + 6 - 2 - \frac{1}{3} - \frac{1}{2} = \frac{6}{3} + 4 - \frac{1}{2} = \frac{16+24-3}{6}$

$\cancel{\frac{8}{3} + 4 = \frac{20}{3}}$ $= 6\frac{1}{2}$ $= \frac{37}{6}$

$= 5\frac{1}{2} = \frac{11}{2}$