Math 231
Final Exam
March 14, 1972

Name *Key*

College________Major________

(30) I. Quick answer. No partial credit. 3 points each.

Correction #2, #4

$\int \frac{1}{\sqrt{x^2-1}}\,dx$

a. $D \cos 3x = -3\sin 3x$

b. $\int_2^3 \frac{1}{x}\,dx = \ln 3 - \ln 2$

c. $\int e^{-x}\,dx = -e^{-x} + C$

d. $\int 4x^5\,dx = \frac{4x^6}{6} + C$

e. $\int \sin x \cos x\,dx = \frac{\sin^2 x}{2} + C$

f. $\int_0^2 \sqrt{4-x^2}\,dx = \frac{\pi}{4}\cdot 4 = \pi$

g. If $y = e^{1/10}$, then $\ln y = \frac{1}{10}$

h. $e^{2\ln 3} = e^{\ln 9} = 9$

i. $D\, e^{3x^2} = e^{3x^2}\cdot 6x$

j. $\ln(\ln e) = 0$

II. Show work for partial credit.

(10) 1. Set up the sum of partial fractions for each of the following rational functions, but do NOT solve for the unknown constants.

a. $\frac{1}{(x-1)^2(x^2+x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Bx+C}{x^2+x+2}$

b. $\frac{1}{(x+2)(x^2+x+2)^2} = \frac{A}{x+2} + \frac{Bx+C}{x^2+x+2} + \frac{Dx+E}{(x^2+x+2)^2}$

c. $\frac{x^4+1}{x^4-1} = 1 + \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$

d. $\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$

$D \ln kx = \frac{1}{kx} \cdot k = \frac{1}{x}$

(5) 2. Differentiate:

a. $f(x) = \sqrt{\ln 3x}$ $\quad \frac{1}{2}(\ln 3x)^{-1/2} \cdot \frac{1}{3x} \cdot 3 = \frac{3}{2} / \sqrt{\ln 3x}$ $\quad \frac{1}{2x\sqrt{\ln 3x}}$

b. $f(x) = \sin x \tan x$ $\quad \sin x \sec^2 x + \cos x \tan x$

$\quad \sin x$

c. $f(x) = 3x\, e^{x^2}$ $\quad 3x\, e^{x^2} \cdot 2x + 3e^{x^2} = (6x^2+3)\, e^{x^2}$

(15) 3. a. Verify: $\int \sec x\, dx = \ln |\sec x + \tan x| + C$

$D \ln|\sec x + \tan x| = \frac{1}{\sec x + \tan x}(\sec x \tan x + \sec^2 x) = \sec x$

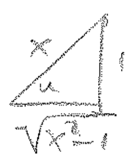
9

b. What, then, is $\int \csc x\, dx$? $= -\ln|\csc x + \cot x| + C$

6 $\quad -\frac{1}{\csc x + \cot x}(-\csc x \cot x - \csc^2 x) = \csc x$

also $\ln|\csc - \cot x| + C$

(10) 4. Use trigonometric substitution to evaluate $\int \frac{1}{\sqrt{x^2-1}}\, dx$.



$x = \csc u$

$dx = -\csc u \cot u\, du$

$\sqrt{x^2-1} = \cot u$

$\int \cot^2 \csc$

$\frac{\cos^2}{\sin^2} \cdot \frac{1}{\sin}$

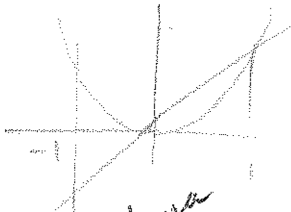
$-\int \frac{1}{\cot u} \csc u \cot u\, du$

$+ \ln|\csc x + \cot x| + C$

$+ \ln\left|\frac{x}{1} + \sqrt{x^2-1}\right| + C$

or $-\ln|x - \sqrt{x^2-1}| + C$

5. Find the area of the region bounded by the straight line $y = x$, the parabola $y = x^2$, and the line $x = -1$.



$$\int_{-1}^{0} x^2 - x \, dx + \int_{0}^{1} x - x^2 \, dx$$

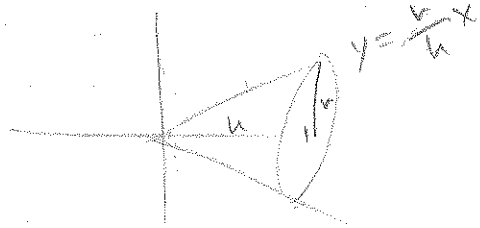
Half area

$$\frac{x^3}{3} - \frac{x^2}{2}\Big|_{-1}^{0} + \frac{x^2}{2} - \frac{x^3}{3}\Big|_{0}^{1} = 1$$

$$0 - \left(-\frac{1}{3} - \frac{1}{2}\right) + \frac{1}{2} - \frac{1}{3} - 0$$

$$= \frac{5}{6} + \frac{1}{6} = \frac{6}{6} = 1$$

(10) 6. Show that the volume of a cone is $V = \frac{1}{3}bh$, where h = height, and b = area of the base.



$$A(x) = \pi r_0^2 = \pi\left(\frac{r}{h}x\right)^2$$

$$= \pi \frac{r^2 x^2}{h^2}$$

$$V = \int_0^h \frac{\pi r^2}{h^2} x^2 \, dx = \frac{\pi r^2}{h^2} \frac{x^3}{3}\Big|_0^h$$

$$= \frac{\pi h^3 r^2}{3h^2} = \frac{\pi r^2 h}{3}$$

III. Evaluate the following integrals: (10 points each)

1. $\displaystyle\int \frac{\cos x}{\sqrt{\sin x}} dx$

$u = \sin x \quad du = \cos x$

$$\int u^{-1/2} du = 2u^{1/2} + C = \frac{3}{4}(\sin x)^{1/2} + C$$

$$2\sqrt{\sin x} + C$$

2. $\int_{-1}^{0} 2x\sqrt{x+1}\, dx$

(i) $u = x+1 \quad x = u-1 \quad dx = du$

$$\int_0^1 2(u-1)\sqrt{u}\, du = \int_0^1 2u^{3/2} - 2u^{1/2}\, du$$

$$= \frac{4}{5}u^{5/2} - \frac{4}{3}u^{3/2}\Big|_0^1 = \frac{4}{5} - \frac{4}{3} - 0 = \frac{-8}{15}$$

3. $\int \frac{1}{x^2+2x+4}\, dx = \int \frac{1}{x^2+2x+1+3}\, dx = \int \frac{1}{(x+1)^2+3}\, dx = \frac{1}{3}\int \frac{1}{\left(\frac{x+1}{\sqrt{3}}\right)^2+1}\, dx$

$u = \frac{x+1}{\sqrt{3}} \quad du = \frac{dx}{\sqrt{3}}$

$$\frac{1}{\sqrt{3}}\int \frac{1}{u^2+1}\, du = \frac{1}{\sqrt{3}}\tan^{-1}u + C = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C$$

4. $\int_0^{\pi/3} \sin 3x \cos^3 3x\, dx = -\frac{1}{3}\int_1^{-1} u^3\, du = -\frac{1}{3}\frac{u^4}{4}\Big|_1^{-1} = -\frac{1}{3}\left[\frac{1}{4} - \frac{1}{4}\right] = 0$

$u = \cos 3x$

$du = -3\sin 3x\, dx$

5. $\int_1^2 \frac{x}{2x^2+5}\, dx = \frac{1}{4}\int_1^2 \frac{4x}{2x^2+5}\, dx = \frac{1}{4}\ln(2x^2+5)\Big|_1^2$

$$= \frac{1}{4}\left[\ln(13) - \ln(7)\right]$$

40

IV. Grading for these questions: Each will be scored 1 to 5. The total number of points will be (4 times the highest score) plus (3 times the next highest)plus(2 times the next)plus(the lowest score). Total,possible = 50. So it is more important to do well on some rather than equally poorly on all. (1 completely right = 20, two = 35 etc.)

1. Describe the process by which the integral of a bounded function is defined, including when it is integrable.

2. Derive the formula for integration by parts.

3. Evaluate: $\displaystyle\int \frac{6x^3 - x^2 + 3x - 2}{(x^2-1)(x^2+2)}\,dx$

4. Evaluate: $\displaystyle\int x^3 \cos 2x\,dx$

2. $Duv = u\,dv + v\,du$

$\int Duv = \int u\,dv + \int v\,du$

$\int u\,dv = uv - \int v\,du$

3. $\dfrac{6x^3 - x^2 + 3x - 2}{(x^2-1)(x^2+2)} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{Cx+D}{x^2+2}$

$6x^3 - x^2 + 3x - 2 = A(x+1)(x^2+2) + B(x-1)(x^2+2)$
$+ (Cx+D)(x^2-1)$

$x = -1$

$-12 = -6 - 1 - 3 - 2 = 0 + B(-2)(3) + 0$

$B = 2$

$x = 1$

$6 - 1 + 3 - 2 = A(2)(3) + 0$

$A = 1$

50

3. (cont) $6x^3 - x^2 + 3x - 2 = (x+1)(x^2+2) + 2(x-1)(x^2+2)$

$+ (Cx+D)(x^2-1)$

$= x^3 + x^2 + 2x + \cancel{2} + 2x^3 - \cancel{2x^2} + 4x - 4$

$+ Cx^3 + Dx^2 - Cx - D$

$= (C+3)x^3 + (D-1)x^2 + (6-C)x - D - 2$

$C + 3 = 6 \qquad D - 1 = -1 \qquad 6 - C = 3 \qquad D = 0$

$C = 3 \qquad D = 0 \qquad C = 3$

$$\int \frac{6x^3 - x^2 + 3x - 2}{(x^2-1)(x^2+2)}\,dx = \int \frac{1}{x-1}\,dx + 2\int \frac{1}{x+1}\,dx + \frac{3}{2}\int \frac{2x}{x^2+2}\,dx$$

$= \ln|x-1| + 2\ln|x+1| + \cancel{\frac{3}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right)} + C$

$\frac{3}{2}\ln(x^2+2)$

4. $\int x^3 \cos 2x\,dx \qquad u = x^3 \quad dv = \cos 2x\,dx$

$du = 3x^2\,dx \quad v = \frac{\sin 2x}{2}$

$= \frac{x^3 \sin 2x}{2} - \frac{3}{2}\int x^2 \sin 2x\,dx \qquad u = x^2 \quad dv = \sin 2x\,dx$

$du = 2x \quad v = -\frac{\cos 2x}{2}$

$\int x^2 \sin 2x\,dx = -\frac{x^2 \cos 2x}{2} + \int x \cos 2x\,dx$

$= -\frac{x^2 \cos 2x}{2} + \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} + C$

$\int x \cos 2x\,dx = \frac{x \sin 2x}{2} - \int \frac{\sin 2x}{2}\,dx \qquad u = x \quad dv = \cos 2x\,dx$

$du = dx \quad v = \frac{\sin 2x}{2}$

$= \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} + C$

(cont)

$$\int x^3 \cos 2x\,dx = \frac{x^3}{2}\sin 2x - \frac{3}{2}\left[-\frac{x^2\cos 2x}{2} + x\frac{\sin 2x}{2} + \frac{\cos 2x}{4}\right] + C$$

$$= \frac{x^3}{2}\sin 2x + \frac{3}{4}x^2\cos 2x - \frac{3}{4}x\sin 2x - \frac{3}{8}\cos 2x + C$$