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Math 231
Final Exam
March 13, 1974 Name KEY

Show work for partial credit!

(6) 1. Define precisely:

a. ln x $= \int_1^x \frac{1}{t}\,dt, \ x > 0$

b. $a^x = e^{x \ln a}$

(45) 2. Short answer, no partial credit.

a. $\int_1^e \frac{1}{x}\,dx = \ln e = 1$

b. $\int 2e^{-3x}\,dx = \frac{2e^{-3x}}{-3} + c$

c. $D\, e^{\pi} = 0$

d. $D\, 2^x = 2^x \ln 2$

e. $\int x^e\,dx = \frac{x^{e+1}}{e+1} + C$

f. $\int \ln 2\,dx = (\ln 2)x + c$

h. $\sin^{-1}(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$

i. $\cos^{-1}(-\frac{1}{2}) = \frac{5\pi}{6}$

j. $\tan(\sin^{-1}\frac{\sqrt{3}}{2}) = \sqrt{3}$

k. $\sin^{-1}(\sin \frac{11}{4}\pi) = \frac{\pi}{4}$

l. $\log_3 x = 2, \ x = 9$

m. $D \ln 2x = \frac{2 \cancel{\ln 2x}}{2x} \ \frac{1}{x}$

n. $D\sqrt{e^x} = e^{x/2} \cdot \frac{1}{2} = \frac{\sqrt{e^x}}{2} = \frac{e^x}{2\sqrt{e^x}}$

o. For what values of x is $y = e^x$ concave up? all x

p. For what values of x is ln x negative ? $0 < x < 1$

3x

x

(15) 6. Partial Fractions:

a. Write each as the sum of partial fractions, do not solve:

i. $\frac{x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$

ii. $\frac{2}{(x-1)^2(x+5)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+5} + \frac{D}{(x+5)^2} + \frac{E}{(x+5)^3}$

iii. $\frac{3x-2}{(2x+5)(x^2+1)^3} = \frac{A}{2x+5} + \frac{Bx+C}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{Ex+F}{(x^2+1)^3}$

b. Find $\int \frac{5x^2+12x+1}{(x-1)(x+2)^2}\,dx = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$

$$5x^2+12x+1 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$$

$x=1$ $\quad 18 = A \cdot 9 + 0$

$A = 2$

$x=-2$ $\quad 20-24+1$

$-3 = 0 + 0 + (-3)C$

$C = 1$

$$5x^2+12x+1 = Ax^2+4Ax+4A + Bx^2+Bx-2B + Cx - C$$
$$= (A+B)x^2 + (4A+B+C)x + 4A-2B-C$$

$A+B = 5$

$4A+B+C = 12$

$4A-2B-C = 1$

$\boxed{A=2 \quad B=3 \quad C=1}$

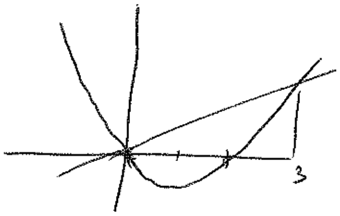
$8+3+1 = 12$ ✓

$8-6-1 = 1$ ✓

$$\int \frac{5x^2+12x+1}{(x-1)(x+1)^2}\,dx = \int \frac{2}{x-1} + \frac{3}{x+2} + \frac{1}{(x+2)^2}\,dx$$

$$= 2\ln|x-1| + 3\ln|x+2| + \frac{(x+2)^{-1}}{-1} + C$$

7. Find the area of the region bounded by the curves $y = x^2-2x$ and $y = x$.



$x = x^2 - 2x$

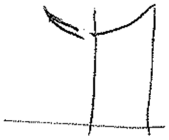
$x^2 - 2x - x = 0$

$x^2 - 3x = 0$

$x(x-3) = 0$

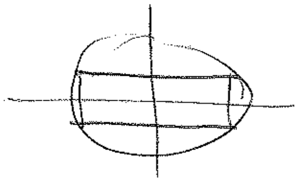
$$A = \int_0^3 \underbrace{x - (x^2 - 2x)}_{3x - x^2}\, dx = \frac{3x^2}{2} - \frac{x^3}{3}\Big|_0^3 = \frac{27}{2} - \frac{27}{3} = \frac{27}{6} = \frac{9}{2}$$

(7) 8. The region bounded by the curves $y = \sec x$, $x = \frac{\pi}{4}$, and the coordinate axes is rotated about the x-axis to form a solid. What is its volume?



$$V = \int_0^{\pi/4} \pi \sec^2 x \, dx = \pi \tan x \Big|_0^{\pi/4} = \pi \cdot 1 - 0 = \pi$$

(7) 9. A 1 inch hole is drilled through the center of a sphere with radius 2 inches. What is the volume of the piece that is left?



$x^2 + y^2 = 4$

$y = \sqrt{4-x^2}$

$$V = \int_{-\sqrt{3}}^{\sqrt{3}} \pi\left(\sqrt{4-x^2}\right)^2 \cancel{dx} - \pi \cdot 1^2 \, dx$$

$y = 1 \quad x = \sqrt{3}$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \pi (4 - x^2 - 1)\, dx = \pi \; 3x - \frac{x^3}{3}\Big|_{-\sqrt{3}}^{\sqrt{3}}$$

$$= 3\sqrt{3} - \frac{3\sqrt{3}}{3} - \left(-3\sqrt{3} + \frac{3\sqrt{3}}{3}\right)$$

$$= 6\sqrt{3} - 2\sqrt{3} = 4\sqrt{3}$$

(7) 10. Integrate each of the following:

a. $\int_1^2 (x+1)\sqrt{2x-1}\,dx =$

1) ~~$u = x+1$~~
$u = 2x-1$
$du = 2x\,dx$
~~$x+1 =$~~
$x = \frac{u+1}{2}$

$\boxed{\frac{1}{2}}\int_{x=1}^{2} \frac{u+3}{2}\sqrt{u}\,du$

$\int_{x=1}^{2} \frac{u^{3/2}}{4} + \frac{3u^{1/2}}{4}\,du$

$\frac{u^{5/2}}{10} + \frac{3u^{3/2}}{2}\Big|_{x=1}^{2} = \frac{(2x-1)^{5/2}}{10} + \frac{(2x-1)^{3/2}}{2}\Big|_1^2 = \frac{3^{5/2}}{10} + \frac{3^{3/2}}{2} - \frac{1}{10} - \frac{1}{2}$

$\frac{9\sqrt{3}}{10} + \frac{3\sqrt{3}}{2} - \frac{6}{10}$

$\frac{24}{10}\sqrt{3} - \frac{6}{10}$

$\boxed{\frac{12}{5}\sqrt{3} - \frac{3}{5}}$

or

$u = x+1 \quad dv = \sqrt{2x-1}\,dx$
$du = dx \quad v = \frac{(2x-1)^{3/2}}{3}$

$= (x+1)\frac{(2x-1)^{3/2}}{3}\Big|_1^2 - \int_1^2 \frac{(2x-1)^{3/2}}{3}\,dx$

$= 4\frac{(3)^{3/2}}{3} - \frac{2}{3} - \frac{(2x-1)^{5/2}}{15}\Big|_1^2$

$= 4\sqrt{3} - \frac{2}{3} - \frac{(3)^{5/2}}{15} - \frac{1}{15} = \left(4 - \frac{9}{15}\right)\sqrt{3} - \frac{16}{15}$

$\frac{51}{15}\sqrt{3} - \frac{16}{15}$

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$\frac{12}{5}\sqrt{3} - \frac{3}{5}$

(7) b. $\int \frac{\cos^3 x}{\sin^2 x}\,dx =$

$\int \frac{1-\sin^2 x}{\sin^2 x}\cos x\,dx$

$u = \sin x$
$du = \cos x\,dx$

$= \int \frac{1-u^2}{u^2}\,du = \int \frac{1}{u^2} - 1\,du$

$= \frac{u^{-1}}{-1} - u + C$

$= -\frac{1}{\sin x} - \sin x + C = -\csc x - \sin x + C$

(7) c. $\int_{-2}^{2} \frac{x^2}{x^2+4}\,dx = \int_{-2}^{2} 1 - \frac{4}{x^2+4}\,dx = \int_{-2}^{2} 1 - \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx$

$$\begin{array}{r} 1 \\ x^2+4 \overline{)\, x^2 } \\ \underline{x^2+4} \\ -4 \end{array}$$

$= x - 2\tan^{-1}\frac{x}{2}\Big|_{-2}^{2}$

$= 2 - 2\tan^{-1} 1 - (-2) + 2\tan^{-1}(-1)$

$= 4 - 2\frac{\pi}{4} - 2\frac{\pi}{4} = 4 - \pi$

(7) d. $\int x^2 \sin x\,dx = -x^2\cos x + \int 2x\cos x\,dx$

$u = x^2 \quad dv = \sin x$
$du = 2x\,dx \quad v = -\cos x$

$= -x^2\cos x + 2x\sin x - \int 2\sin x\,dx$

$= -x^2\cos x + 2x\sin x + 2\cos x + C$

$u = 2x \quad dv = \cos x\,dx$
$du = 2\,dx \quad v = \sin x$

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$-2x-1 \qquad x = \frac{u+1}{2} \qquad x+1 = \frac{u+3}{2}$

$du = 2dx$

$$\frac{1}{2}\int_{x=1}^{2} \frac{u+3}{2}\sqrt{u}\,du = \frac{1}{4}\int_{x=1}^{2} u^{3/2} + 3u^{1/2}\,du$$

$$= \frac{1}{4}\frac{u^{5/2}}{5/2} + {}_4\frac{3u^{3/2}}{3/2}\Big|_{x=1}^{2}$$

$$= \frac{(2x-1)^{5/2}}{10} + \frac{21(2x-1)^{3/2}}{\cancel{4}\,2}\Big|_{x=1}^{2}$$

$$= \frac{9\sqrt{3}}{10} + \cancel{2}\cdot\frac{3\sqrt{3}}{2} - \frac{1}{10} - \frac{\cancel{1}}{2}$$

$$= ~~\frac{49}{10}\sqrt{3} - \frac{21}{10}~~$$

$$= \frac{(9+15)\sqrt{3}}{10} - \frac{6}{10} = \frac{24\sqrt{3}}{10} - \frac{6}{10}$$

$u = x+1 \qquad dv = \sqrt{2x-1}\,dx$

$du = dx \qquad v = \frac{(2x-1)^{3/2}}{3}$

$$(x+1)\frac{(2x-1)^{3/2}}{3}\Big|_1^2 - \int_1^3 \frac{(2x-1)^{3/2}}{3}\,dx$$

$$\frac{3}{3}(3)^{3/2} - \frac{2}{3} - \frac{(2x-1)^{5/2}}{15}\Big|_1^{2}$$

$$= \overset{3}{\cancel{3}}\sqrt{3} - \frac{2}{3} - \frac{9\sqrt{3}}{15} + \frac{1}{15}$$

$$= ~~\frac{36}{15}\sqrt{3} - \frac{14}{15}~~$$

$\frac{36}{15}$

$\frac{36\sqrt{3}}{15} + \frac{-10+1}{15}$

$\frac{12\sqrt{3}}{5} + \frac{-9}{15}$

$\frac{12\sqrt{3}}{5} - \frac{3}{5}$

$3\frac{3}{5} = \frac{15+3}{5} = \frac{18}{5}$

$e^{x/2}$

$3^2 = x$

$\frac{1}{\sqrt{2}}$