MATH 231
TEST I
January 18, 1974

NAME KEY

SHOW WORK FOR PARTIAL CREDIT!

(5) 1. State precisely the Fundamental Theorem of Calculus.

If f is continuous on $[a,b]$, and $F'(x)=f(x)$ for x in $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b)-F(a).$$

(15) 2. (a) Write out the upper sum $T(P)$ for the function $f(x) = \frac{3}{x} + 1$ on $[1,10]$ using partition $P = \{\underbrace{1,2}_{1},\underbrace{2,4}_{2},\underbrace{4,6}_{2},\underbrace{6,7}_{1},\underbrace{7,10}_{3}\}$.

$\searrow$ LHS

$$T(P) = f(1)\cdot 1 + f(2)\cdot 2 + f(4)\cdot 2 + f(6)\cdot 1 + f(7)\cdot 3$$
$$= \left(\tfrac{3}{1}+1\right)\cdot 1 + \left(\tfrac{3}{2}+1\right)\cdot 2 + \left(\tfrac{3}{4}+1\right)\cdot 2 + \left(\tfrac{3}{6}+1\right)\cdot 1 + \left(\tfrac{3}{7}+1\right) 3$$
$$= 4\cdot 1 + \tfrac{5}{2}\cdot 2 + \tfrac{7}{4}\cdot 2 + \tfrac{3}{2}\cdot 1 + \tfrac{10}{7}\cdot 3$$
$$= 4 + 5 + \tfrac{7}{2} + \tfrac{3}{2} + \tfrac{30}{7} = 14 + \tfrac{30}{7} = \tfrac{98+30}{7} = \tfrac{128}{7}$$

(b) Write out the lower sum $S(P)$ for $f(x) = 3x^2 - 12$ on $[-3,4]$ using the partition $P = \{\underbrace{-3,-2}_{1},\underbrace{-2,-1}_{1},\underbrace{-1,1}_{2},\underbrace{1,3}_{2},\underbrace{3,4}_{1}\}$.

$f'(x) = 6x$

$$S(P) = \left(3(-2)^2-12\right)\cdot 1 + \left(3(-1)^2-12\right)\cdot 1 + \left(3(0)^2-12\right)\cdot 2 + \left(3(1)^2-12\right)\cdot 2 + \left(3(3)^2-12\right)\cdot 1$$
$$= 0\cdot 1 + (-9)\cdot 1 + (-12)(2) + (-9)\cdot 2 + 15\cdot 1$$
$$= 0 - 9 - 24 - 18 + 15 = -36$$

(10) 3. Write out:

(a) $\sum_{i=1}^{6} \frac{i+1}{2+i} = \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \frac{6}{7} + \frac{7}{8}$

(b) $\sum_{k=5}^{8} k2^{k-1} = 5\cdot 2^4 + 6\cdot 2^5 + 7\cdot 2^6 + 8\cdot 2^7$

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(10) 4. True or false? Why?

(a) $\int (3-x)^2 dx = \frac{(3-x)^3}{3} + C$

$$D\,\frac{(3-x)^3}{3} = \frac{3(3-x)^2(-1)}{3} = -(3-x)^2$$

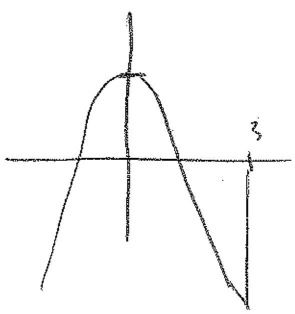
False

(b) $\int \frac{x^2}{x+1}\, dx = \frac{x^3}{x+1} + C$

$$D\,\frac{x^3}{x+1} = \frac{(x+1)(3x^2) - x^3(1)}{(x+1)^2} = \frac{3x^3+3x^2-x^3}{(x+1)^2}$$

$$= \frac{2x^3+3x^2}{(x+1)^2} \quad \text{NO}$$

(20) 5. Consider $f(x) = 1 - x^2$ on $[0,3]$. Compute the upp sum $T(P_n)$ for the regular partition P_n, so that $[0,3]$ is divided up into n equal subintervals, and find $\lim_{n\to\infty} T(P_n)$.



$$[0,3] \quad \left\{0, \frac{3}{n}, \frac{6}{n}, \dots, \frac{3n}{n}\right\}$$

$$x_i = \frac{3i}{n} \qquad x_{i-1} = \frac{3(i-1)}{n}$$

$$T(P_n) = \sum_{i=1}^{n}\left[1 - \left(\frac{3(i-1)}{n}\right)^2\right]\cdot\frac{3}{n}$$

$$= \sum_{i=1}^{n}\frac{3}{n} - \frac{27}{n^3}\sum_{i=2}^{n}(i-1)^2$$

$$= 3 - \frac{27}{n^3}\sum_{j=1}^{n-1} j^2$$

$$= 3 - \frac{27}{n^3}\,\cancel{\sum}\,\frac{(n-1)(n)(\overset{2n-1}{2(n-1)+1})}{6}$$

$$= 3 - \frac{27}{6}\,\frac{n-1}{n}\cdot\frac{n}{n}\cdot\frac{2n-1}{n}$$

$$= 3 - \frac{9}{2}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$$

$$\lim_{n\to\infty} T(P_n) = 3 - 9 = -6$$

30 $\quad \int_0^3 1-x^2\,dx = x - \frac{x^3}{3}\Big|_0^3 = 3 - \frac{27}{3} = 3-9=-6$

(30) 6. Compute the following intervals

(a) $\int_0^1 x^4 - 3x^2 + 5dx =$ $\frac{x^5}{5} - \frac{3x^3}{3} + 5x \Big|_0^1$

$= \frac{1}{5} - 1 + 5 - 0 = 4 + \frac{1}{5} = 4\frac{1}{5} = \frac{21}{5}$

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(b) $\int_1^4 x^2 - \sqrt{x} + \frac{4}{x^2}\, dx =$ $\int_1^4 x^2 - x^{1/2} + 4x^{-2} dx$

$= \frac{x^3}{3} - \frac{x^{3/2}}{3/2} + \frac{4x^{-1}}{-1} \Big|_1^4 = \frac{4^3}{3} - \frac{2}{3}(4)^{3/2} - \frac{4}{4} - \left(\frac{1}{3} - \frac{2}{3} - 4\right)$

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$= \frac{64}{3} - \frac{16}{3} - 1 + \frac{1}{3} + 4$

$= \frac{48}{3} + 3 + \frac{1}{3} = 19\frac{1}{3}$

(c) $\int_0^2 (3x-4)^{10} dx =$ $\frac{1}{3}\int_0^2 (3x-4)^{10} 3dx$

$= \frac{1}{3}\frac{(3x-4)^{11}}{11}\Big|_0^2 = \frac{1}{3}\frac{(6-4)^{11}}{11} - \frac{1}{3}\frac{(-4)^{11}}{11}$

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$= \frac{2^{11}}{33} + \frac{4^{11}}{33}$

(d) $\int \frac{x}{\sqrt{x^2-6}}\, dx =$ $\int (x^2-6)^{-1/2} x\, dx = \frac{1}{2}\int (x^2-6)^{-1/2} 2x\, dx$

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$= \frac{1}{2}\frac{(x^2-6)^{1/2}}{1/2} + C = (x^2-6)^{1/2} + C$

$7\frac{1}{2}$

$= \sqrt{x^2-6} + C$

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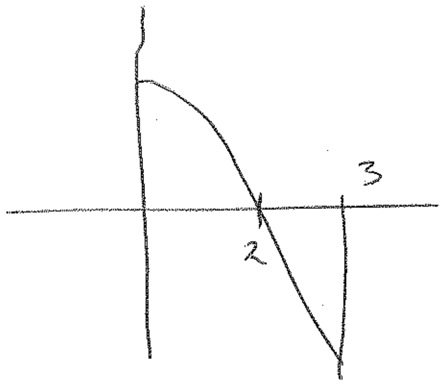
7. Find the area of the region bounded by the curves $y = 4 - x^2$, x-axis, $x = 0$, and $x = 3$.



$$\text{area} = \int_0^2 4 - x^2\,dx - \int_2^3 4 - x^2\,dx$$

$$= 4x - \frac{x^3}{3}\Big|_0^2 - \left[4x - \frac{x^3}{3}\Big|_2^3\right]$$

$$= 8 - \frac{8}{3} - 0 - \left(12 - \overset{9}{\overset{\|}{\frac{27}{3}}}\right) + 8 - \frac{8}{3}$$

$$= 16 - 3 - \frac{16}{3} = 13 - \frac{16}{3}$$

$$= \frac{39 - 16}{3} = \frac{23}{3} = 7\frac{2}{3}$$

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