Math 231
Final Exam
March 21, 1975

Name KEY

8:00
25

doing all prob

I. Short answer. 5 points each.

1. $\int e^{-x}\,dx = -e^{-x} + C$

2. $\int_1^2 \frac{1}{x+1}\,dx = \ln|x+1|\Big|_1^2 = \ln 3 - \ln 2$

3. $\int \frac{2x^3 - x^2}{(1+x)^2}\,dx = \frac{x^2}{1+x} + C$ True or false? Why?

$D\left(\frac{x^2}{1+x}\right) = \frac{(1+x)2x - x^2\cdot 1}{(1+x)^2} = \frac{2x + 2x^2 - x^2}{(1+x)^2} = \frac{2x + x^2}{(1+x)^2}$ FALSE

4. If $F(x) = \int_1^{x^2} e^{-t^2}\,dt$, then $F'(x) = e^{-(x^2)^2}\cdot 2x = 2x\,e^{-x^4}$

5. Write out the form (ONLY) of the partial fractions expansion for $\frac{x^2}{x^2(x-1)(x^2+2)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+2} + \frac{Ex+F}{(x^2+2)^2}$

DO NOT SOLVE!

6. $\int \sin 3x\,dx = -\frac{\cos 3x}{3} + C$

7. $\int 3(1+\sec^2 x)\,dx = 3x + 3\tan x + C$

8. $\int \frac{x}{x^2+5}\,dx = \frac{1}{2}\ln(x^2+5) + C$

9. $\frac{d}{dx}\sec(x^2+2) = \sec(x^2+2)\tan(x^2+2)\cdot 2x$

10. $\frac{d}{dx}\ln(e^x+1) = \frac{1}{e^x+1}\cdot e^x$

I. cont.

11. $\frac{d}{dx}\csc(2x+3) =$ $-\csc(2x+3)\cot(2x+3)\cdot 2$

12. $\frac{d}{dx}\sec^2 x =$ $2\sec x \sec x \tan x = 2\sec^2 x \tan x$

13. $\frac{d}{dx}\sqrt{\sin^{-1} x} =$ $\frac{1}{2}(\sin^{-1}x)^{-1/2}\frac{1}{\sqrt{1-x^2}} = \frac{1}{2\sqrt{\sin^{-1}x}\sqrt{1-x^2}}$

14. DEFINE $y = e^x$ if and only if $x = \ln y$

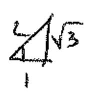
15. If $\frac{dy}{dx} = y$ and $y = 2$ when $x = 1$, then $y =$ $\frac{2}{e}e^x = 2e^{x-1}$

$y = ce^x$ $\quad c = \frac{2}{e}$

$2 = ce^1$

16. $\frac{d}{dx}3^x =$ $\frac{d}{dx}e^{x\ln 3} = e^{x\ln 3}\cdot\ln 3 = 3^x\ln 3$

17. $\tan^{-1}(-\sqrt{3}) =$ $-\frac{\pi}{3}$



II. Problems: Omit one of the 15 point questions by crossing it out!

(15) 1. Evaluate these integrals or show that they diverge:

a. $\int_0^2 \frac{3}{x^2}\,dx = \lim_{a\to 0^+}\int_a^2 3x^{-2}\,dx = \lim_{a\to 0^+}\left.\frac{3x^{-1}}{-1}\right|_a^2 = \lim_{a\to 0^+}\left(-\frac{3}{2}+\frac{3}{a}\right)$

$= \infty$ DIV

b. $\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_1^b e^{-x}\,dx = \lim_{b\to\infty}\left.-e^{-x}\right|_1^b = \lim_{b\to\infty} -\underset{\searrow 0}{e^{-b}} + e^{-1} = e^{-1}$

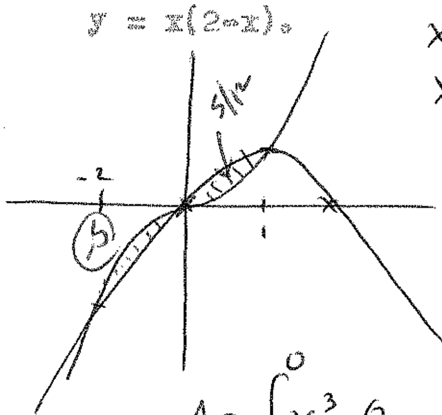
c. $\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\left.\tan^{-1}x\right|_0^b$

$= \lim_{b\to\infty}\tan^{-1} b - 0 = \frac{\pi}{2}$

50 a 35

II. cont.

(10) 2. Find the area of the region bounded by the curves $y = x^3$ and $y = x(2-x)$.



$x^3 = x(2-x) = 2x - x^2$

$\cancel{x^3+2x}$ $x^3 + x^2 - 2x = 0$

$x(x^2+x-2) = 0$

$x(x-1)(x+2) = 0$

$x = 0, 1, -2$

$$A = \int_{-2}^{0} x^3 - \underbrace{(2x - x^2)}_{-2x+x^2} \, dx + \int_0^1 2x - x^2 \, dx - x^3 \, dx$$

$$= \frac{x^4}{4} - x^2 + \frac{x^3}{3}\Big|_{-2}^{0} + x^2 - \frac{x^3}{3} - \frac{x^4}{4}\Big|_0^1 = -\frac{16}{4} + 4 - \frac{(-8)}{3} + 1 - \frac{1}{3} - \frac{1}{4}$$

$\frac{8}{3} + 1 - \frac{1}{3} - \frac{1}{4}$

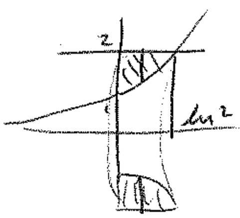
$\frac{28+12-3}{12} = \frac{37}{12}$

$= -2 + \frac{8}{3} + 1 - \frac{1}{3} - \frac{1}{4}$

$= -1 - \frac{1}{4} + \frac{7}{3} = \frac{-12-3+28}{12}$

$\frac{13}{12}$

(15) 3. The region bounded by the curves $y = e^x$, $y = 2$ and the y-axis is revolved about the x-axis. Find the volume of the solid generated.



$A(x) = \pi 2^2 - \pi (e^x)^2$

$$V = \int_0^{\ln 2} 4\pi - e^{2x}\pi \, dx = 4\pi x - \frac{e^{2x}\pi}{2}\Big|_0^{\ln 2}$$

$$= 4\pi \ln 2 - 2\pi - 0 + \frac{\pi}{2}$$

$$= \pi\left(4\ln 2 - \frac{3}{2}\right)$$

$\ln y$ $2\pi y$

OR SHELLS

$$\int_1^2 2\pi y \ln y \, dy = 2\pi\left[\frac{y^2}{2}\ln y \Big|_1^2 - \int_0^1 \frac{y}{2} dy\right] = 2\pi\left[\frac{4}{2}\ln 2 - \frac{1}{2}\ln 1 - \frac{y^2}{4}\Big|_1^2\right]$$

$u = \ln y$ $dv = y\,dy$

$du = \frac{1}{y}dy$ $v = \frac{y^2}{2}dy$

$$= 2\pi\left[2\ln 2 - \cancel{1} + \frac{1}{4}\right] = 4\pi\ln 2 - \frac{3}{2}\pi$$

(5) 4. (Set up, do not integrate) Find the arc length of the curve $y = \sin^{-1}x$.

$y' = \frac{1}{\sqrt{1-x^2}}$

$$L = \int_{(-1)}^{1} \sqrt{1 + \left(\frac{1}{\sqrt{1-x^2}}\right)^2} \, dx$$

$\cancel{\frac{1}{\sqrt{2}}}$

$\cancel{0}$

25 or 20

(15) 5. Suppose the population of a certain country is growing at a rate proportional to the population, and it can only feed 20,000,000 each year. In 1940 there were 1,000,000 people and in 1960 there were 5,000,000.

a. What is the population in 1975?

b. When will the country no longer be able to feed itself? (During which year?)

ln 2= .7
ln 3=1.1
ln 5=1.6
ln 7=1.9
ln11=2.4
ln13=2.6
ln17=2.8
ln19=2.9
ln23=3.1
ln29=3.4
ln31=3.4
ln37=3.6

$y = Ce^{kt}$ $t = 0$ is 1940

$C = 1{,}000{,}000$

$5{,}000{,}000 = 1000000 e^{20k}$

$20k = \ln 5$

$k = \frac{1}{20}\ln 5$

$y = 1000000\, e^{\frac{1}{20}\ln 5\, t}$

$t = 35$

$y = 1{,}000{,}000\, e^{\frac{35}{20}\ln 5}$ $\quad \frac{7}{4}$

$= 1{,}000{,}000\ 5^{7/4}$

$= 17{,}000{,}000 \quad e^{2.8} = 17$

$\frac{1.6 \cdot 7}{4} = 28$

$1000000\, e^{\frac{1}{20}\ln 5 t} = 20{,}000{,}000$

$\frac{1}{20}\ln 5\, t = \ln 20$

$t = 20\frac{\ln 20}{\ln 5}$

$= \frac{20 \cdot 3}{1.6} = \frac{60}{1.6}$

$= 37\frac{1}{2}$

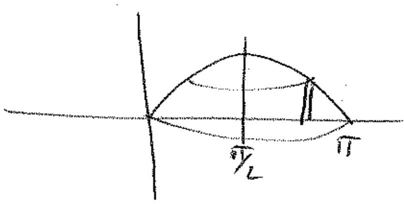
$\boxed{1977}$

$\ln 20 = 2\ln 2 + \ln 5$

$= 1.4 + 1.6 = 3$

$1.6 \overline{)600.}$ → $37.\frac{1}{2}$; 48, 120, 112, 80

(15) 6. The region bounded by the curve $y = \sin x$, $0 \le x \le \pi$, and the x-axis is revolved around the line $x = \frac{\pi}{2}$. Find the volume of this solid. ~~(extra credit: If a hole with diameter π is drilled out of the center, how much is left?)~~



shells

$\int_{\pi/2}^{\pi} \sin x\ 2\pi\left(x - \frac{\pi}{2}\right) dx$

$u = x - \frac{\pi}{2}$ $\quad dv = \sin x$

$du = dx$ $\quad v = -\cos x$

$= 2\pi\left(-\cos x\left(x - \frac{\pi}{2}\right)\Big|_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} -\cos x\, dx\right)$

$= 2\pi\left(-\cos x\left(x - \frac{\pi}{2}\right)\Big|_{\pi/2}^{\pi} + \sin x\Big|_{\pi/2}^{\pi}\right)$

$= 2\pi\left(-(-1)\left(\frac{\pi}{2}\right) + 0 + 0 - 1\right)$

$= 2\pi\left(\frac{\pi}{2} - 1\right) = \pi^2 - 2\pi$

30 of 15

III. DO ~~FIVE~~ FOUR ~~12~~ $12\frac{1}{2}$ points each. Cross out the one omitted.

1. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos^2 x + 1}\, dx = -\int_1^0 \frac{1}{u^2+1}\, du = -\tan^{-1} u \Big|_1^0 = -0 + \frac{\pi}{4} = \frac{\pi}{4}$

$u = \cos x$

$du = -\sin x\, dx$

2. $\displaystyle\int_0^1 (x-2)\sqrt{x+1}\, dx = \int_{x=1}^{2} (u-3)\sqrt{u}\, du = \int_1^2 u^{3/2} - 3u^{1/2}\, du = \frac{u^{5/2}}{5/2} - \frac{3u^{3/2}}{3/2}\Big|_1^2$

$u = x+1 \quad x = u-1$

$du = dx$

$= \frac{2}{5}2^{5/2} - \frac{2}{\cancel{3}}2^{3/2} - \frac{2}{5} + \frac{2}{\cancel{3}}$

$= \frac{8\sqrt{2} - 2}{5} - \frac{4\sqrt{2} - 2}{\cancel{3}}$

$= \frac{2}{5}\sqrt{32} - 2\sqrt{8} + \frac{8}{5}$

$= \frac{8}{5}\sqrt{2} - 4\sqrt{2} + \frac{8}{5}$

$= \frac{(8 - 20\sqrt{2} + 8}{5} = \frac{8 - 12\sqrt{2}}{5}$

$u = x-2 \quad dv = \sqrt{x+1}\, dx$

$du = dx \quad v = \frac{2}{3}(x+1)^{3/2}$

$= (x-2)\frac{2}{3}(x+1)^{3/2}\Big|_0^1 - \int_0^1 \frac{2}{3}(x+1)^{3/2}\, dx$

$= \frac{2}{3}(x-2)(x+1)^{3/2}\Big|_0^1 - \frac{4}{15}(x+1)^{5/2}\Big|_0^1$

$= \frac{2}{3}(-1)2^{3/2} - \frac{2}{3}(-2) - \frac{4}{15}2^{5/2} + \frac{4}{15} = -\frac{4\sqrt{2}}{3} + \frac{4}{3} - \frac{16\sqrt{2}}{15} + \frac{4}{15}$

$= \frac{-20\sqrt{2} - 20 - 16\sqrt{2} + 4}{15} = \frac{-36\sqrt{2} + 24}{15} = -\frac{12\sqrt{2}}{5} + \frac{8}{5}$

3. $\displaystyle\int \frac{1}{x^2 - 2x + 5}\, dx = \int \frac{1}{(x-1)^2 + 4}\, dx = \int \frac{1}{u^2+4}\, du = \frac{1}{4}\int \frac{1}{(\frac{u}{2})^2 + 1}\, du$

$(x^2 - 2x + 1) + 4$

$u = x-1$

$du = dx$

$= \frac{1}{2}\tan^{-1}\frac{u}{2} + C$

$= \frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C$

20 or 30

III.

4. $$\int x \ln x \, dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2x}\,dx = \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

$u = \ln x \quad dv = x$

$du = \frac{1}{x}dx \quad v = \frac{x^2}{2}$

5. $$\int \frac{x^3}{(1+x^2)^3}\,dx = \frac{1}{2}\int \frac{u-1}{u^3}\,du = \frac{1}{2}\int u^{-2} - u^{-3}\,du$$

$u = (1+x^2) \quad x^2 = u-1$

$du = 2x$

$$= \frac{1}{2}\left(\frac{u^{-1}}{-1} - \frac{u^{-2}}{-2}\right) + C = -\frac{1}{2(1+x^2)} + \frac{1}{4(1+x^2)^2}$$

OR $\quad u = x^2 \quad dv = \frac{x}{(1+x^2)^3}dx$

$du = 2x\,dx \quad v = \frac{1}{2}\frac{(1+x^2)^{-2}}{-2}$

$= -\frac{1}{4(1+x^2)^2}$

$$= -\frac{x^2}{4(1+x^2)^2} - \int \frac{-2x}{4(1+x^2)^2}\,dx$$

$$= \frac{-x^2}{4(1+x^2)^2} + \frac{1}{4}\frac{(1+x^2)^{-1}}{-1} + C$$

$$= \frac{-x^2}{4(1+x^2)^2} - \frac{1}{4(1+x^2)} + C$$

6. $$\int \frac{1}{(x-1)(x+2)^2}\,dx = \int \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}\,dx = \int \frac{1/9}{x-1} + \frac{-1/9}{x+2} + \frac{+1/3}{(x+2)^2}\,dx$$

$$= \frac{1}{9}\ln|x-1| - \frac{1}{9}\ln|x+2| - \frac{1}{3}\frac{(x+2)^{-1}}{-1} + C$$

$1 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$

$x = 1$

$1 = 9A \quad A = \frac{1}{9}$

$B = \frac{4+3-9}{18} = -\frac{2}{18} = -\frac{1}{9}$

$x = -2$

$1 = -3C \quad C = -\frac{1}{3}$

$x = 0 \quad 1 = 4A - 2B - C$

$1 = \frac{4}{9} - 2B + \frac{1}{3}$

$2B = \frac{4}{9} + \frac{1}{3} - 1$

$$= \frac{1}{9}\ln|x-1| - \frac{1}{9}\ln|x+2| + \frac{1}{3(x+2)} + C$$