$\frac{11{:}1\frac{12}{\cancel{18}}}{6}$

Math 231
Test III
March 7, 1975

Name KEY

I+II $1\cancel{8}$

III $\frac{6}{12}$

I. 5 pts. Write out(ONLY) the partial fractions expansion of each of the following expressions. Do NOT solve.

a. $\dfrac{x}{(x+1)^2(x^2+4)} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{Cx+D}{x^2+4}$

b. $\dfrac{2}{x(x+1)(x-1)^3} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1} + \dfrac{D}{(x-1)^2} + \dfrac{E}{(x-1)^3}$

II. Required method 15 pts.ea.

$u = 1+\sqrt{x}$

$du = \frac{1}{2\sqrt{x}}dx$

1. Change of variable:

$$\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx = \int \frac{z}{1+z}\,2z\,dz = 2\int \frac{z^2}{z+1}\,dz$$

$z^2 = x$

$2z\,dz = dx$

$$= 2\int z - 1 + \frac{1}{z+1}\,dz$$

$$= 2\left[\frac{z^2}{2} - z + \ln|z+1|\right] + C$$

$$= +x - 2\sqrt{x} + 2\ln|\sqrt{x}+1| + C$$

$$\begin{array}{r} z-1 \\ z+1\,\overline{)\,z^2} \\ \underline{z^2+z} \\ -z \\ \underline{-z-1} \\ 1 \end{array}$$

or $u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}dx$

$$2\int \frac{u^2}{1+u}\,du = 2\int u - 1 + \frac{1}{u+1} = \frac{u^2}{2} - u + \ln|u+1| + C$$

$$\begin{array}{r} u+1 \\ u+1\,\overline{)\,u^2} \\ \underline{u^2+u} \\ \frac{u}{u+1} \end{array}$$

$$= \frac{(\cancel{1+}\sqrt{x})^2}{\cancel{2}} - \frac{2}{}(\cancel{1+}\sqrt{x}) + \ln|1+\sqrt{x}| + C$$

$$= \frac{x}{\cancel{2}} - 2\sqrt{x} + 2\ln|1+\sqrt{x}| + C$$

2. Integration by parts

$$\int 2x\,e^{-x}\,dx = -2xe^{-x} - \int -2e^{-x}\,dx$$

$u = 2x \quad dv = e^{-x}dx$

$du = 2\,dx \quad v = -e^{-x}$

$$= -2xe^{-x} + 2(-e^{-x}) + C$$

$$= -2xe^{-x} - 2e^{-x} + C$$

$$= -2e^{-x}(x+1) + C$$

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II. 3. Partial fractions:

$$\int \frac{2x}{(x-1)(x^2+1)}\,dx = \int \frac{1}{x-1} + \frac{-x+1}{x^2+1}\,dx$$

$$\frac{2x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$2x = A(x^2+1) + (Bx+C)(x-1)$$

$$\cancel{x}\,2x = Ax^2 + A + Bx^2 + Cx - Bx - C$$

$$= (A+B)x^2 + (C-B)x + A - C$$

$A + B = 0$

$C - B = 2$

$A - C = 0$

$x = 1$

$2 = 2A$

$A = 1$

$C = 1$

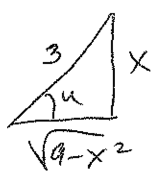
$B = -1$

$$\int \frac{1}{x-1} - \frac{x}{x^2+1} + \frac{1}{x^2+1}\,dx$$

$$\ln|x-1| - \frac{1}{2}\ln|x^2+1| + \tan^{-1}x + C$$

4. Trigonometric substitution:

$$\int \frac{1}{(9-x^2)^{3/2}}\,dx = \int \frac{1}{(3\cos u)^3}\,3\cos u\,du = \int \frac{1}{9\cos^2 u}\,du$$



$\frac{x}{3} = \sin u$

$dx = 3\cos u\,du$

$\sqrt{9-x^2} = 3\cos u$

$$= \frac{1}{9}\int \sec^2 u\,du$$

$$= \frac{1}{9}\tan u + C$$

$$= \frac{1}{9}\,\frac{x}{\sqrt{9-x^2}} + C$$

5. $\int x^3 \sqrt{1-x^2}\, dx \quad = -\frac{1}{2}\int (1-u)\ \sqrt{u}\ du$

$u = 1-x^2$ $\qquad = -\frac{1}{2}\int u^{1/2} - u^{3/2}\, du = -\frac{1}{2}\left[\frac{u^{3/2}}{3/2} - \frac{u^{5/2}}{5/2}\right] + C$

$du = -2x\, dx$

$= -\frac{1}{3}(1-x^2)^{3/2} + \frac{1}{5}(1-x^2)^{5/2} + C$

$u = x^2 \qquad dv = (1-x^2)^{1/2}\, x\, dx$

$du = 2x\, dx \qquad v = \frac{\frac{2}{3}(1-x^2)^{3/2}}{-2} = -\frac{1}{3}(1-x^2)^{3/2}$

$-\frac{x^2(1-x^2)^{3/2}}{3} - \int -\frac{1}{3}(1-x^2)^{3/2}\, 2x\, dx$

$-\frac{x^2(1-x^2)^{3/2}}{3} - \frac{2}{15}(1-x^2)^{5/2} + C$

3. $\int \tan^3 x \sec^2 x \, dx = \int u^3 \, du = \frac{u^4}{4} + C = \frac{\tan^4 x}{4} + C$

$u = \tan x$

$du = \sec^2 x \, dx$

$$\int \frac{x}{x^2+4x+8}\,dx = \int \frac{x}{x^2+4x+4+4}\,dx = \int \frac{x}{(x+2)^2+4}\,dx$$

$u = x+2$

$du = dx$

$$= \int \frac{u-2}{u^2+4}\,du = \int \frac{u}{u^2+4}\,du - 2\int \frac{1}{u^2+4}\,du$$

$$= \frac{1}{2}\ln|u^2+4| - \frac{2}{4}\int \frac{1}{(\frac{u}{2})^2+1}\,du$$

$$= \frac{1}{2}\ln|u^2+4| - \int \frac{1/2}{(\frac{u}{2})^2+1}\,du$$

$$= \frac{1}{2}\ln|u^2+4| - \tan^{-1}\left(\frac{u}{2}\right) + C$$

$$= \frac{1}{2}\ln|x^2+4x+8| - \tan^{-1}\left(\frac{x+2}{2}\right) + C$$

$$\int \frac{\sin x}{1+\cos x}\,dx = -\int \frac{1}{1+u}\,du = -\ln|1+u| + C$$

$$= -\ln|1+\cos x| + C$$

$u = \cos x$

$du = -\sin x\,dx$

III. Free style
a. 10 points if done in class.
b. 5 points if done at home-turn in Monday.

1. $\int \frac{x}{\sqrt{4-x^2}}\,dx = -\frac{1}{2}\int u^{-1/2}\,du = -\frac{1}{2}\frac{u^{1/2}}{1/2} + C$

$= -\sqrt{4-x^2} + C$

$u = 4 - x^2$

$du = -2x\,dx$