Math 232
Final Exam
Dec. 8, 1969

Name Key

Show all work. Each problem worth 10 points.

1. Let point A be (1,2,0), B (3,0,4), and C (2,1,1).

a) What is the cosine of the angle ABC ?

$\cancel{AB}\ \vec{BA} = -2i + 2j - 4k$ $\qquad |\vec{BA}| = \sqrt{4+4+16} = \sqrt{24}$

$\vec{BC} = -i + j - 3k$ $\qquad |\vec{BC}| = \sqrt{1+1+9} = \sqrt{11}$

$\vec{BA} \cdot \vec{BC} = 2 + 2 + 12 = 16$ $\qquad \cos\theta = \dfrac{16}{\sqrt{24}\sqrt{11}} = \dfrac{8}{\sqrt{6}\sqrt{11}}$

b) Is $\vec{AB}$ perpendicular to $\bar{I} + \bar{J}$?

$\vec{AB} = 2i - 2j + 4k$

$\vec{AB} \cdot (i + j) = 2 - 2 = 0$

yes

2. Find the equation(s) of the line through (1,2,-2) parallel to the line through (10,9,7) and (12,3,4).

$\vec{B} = i + 2j - 2k$

$\vec{M} = 2i - 6j - 3k$

$\vec{F}(t) = (2i - 6j - 3k)t + i + 2j - 2k$

$= (2t+1)i + (-6t+2)j + (-3t-2)k$

3. What is the length of the arc of the vector equation

$\bar{F}(t) = t^3\,\bar{I} + 2t^2\,\bar{J} + \bar{K}$

between the points (-8,8,1) and (1,2,1) ?

$t=-2 \qquad t=1$

$\vec{F}'(t) = 3t^2 i + 4t j + 0k$

$|F'(t)| = \sqrt{9t^4 + 16t^2} = t\sqrt{9t^2+16}$

$u = 9t^2 + 16$

$du = 18t\,dt$

$\frac{1}{18}\int_{-2}^{1} 18t\sqrt{9t^2+16}\,dt$

$= \frac{2}{3 \cdot 18}(9t^2+16)^{3/2}\Big|_{-2}^{1} = \dfrac{125 - (52)^{3/2}}{3 \cdot 9}$

4. Find the unit tangent vector to the curve

$$\bar{F}(t) = e^t \bar{i} + t^2 \bar{j} + (t+1)\bar{k}$$

at the point $(\frac{1}{e}, 1, 0)$.

$$\vec{F}'(t) = e^t \bar{i} + 2t\bar{j} + \bar{k}$$

$$F'(-1) = e^{-1}\bar{i} - 2\bar{j} + \bar{k}$$

$$|F'(-1)| = \sqrt{e^{-2}+5}$$

$$\vec{T} = \frac{e^{-1}}{\sqrt{e^{-2}+5}}\,\bar{i} - \frac{2}{\sqrt{e^{-2}+5}}\,\bar{j} + \frac{1}{\sqrt{e^{-2}+5}}$$

t

5. Find equation(s) of the curve of intersection of the planes

$$y = x + 1, \quad \text{and} \quad x + y + z = 1.$$

$x = t \quad y = t + 1$

$t + t + 1 + z = 1$

$z = -2t$

$x = t \quad y = t + 1 \quad z = -2t$

6. Find the equations(s) of the line tangent to the helix

$$\bar{F}(t) = 2\sin t\,\bar{i} + 2\cos t\,\bar{j} + 3t\bar{k}$$

at the point where the helix intersects the x-y plane.

$$\vec{F}'(t) = 2\cos t\,\bar{i} - 2\sin t\,\bar{j} + 3k$$

$$\vec{M} = \vec{F}'(0) = 2\bar{i} + 3k \qquad \vec{B} = \vec{F}(0) = 2\bar{j}$$

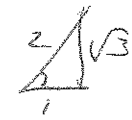
$$G(s) = s(2\bar{i} + 3k) + 2\bar{j}$$

$$= 2s\bar{i} + 2\bar{j} + 3sk$$

7. $2x^2 + 2\sqrt{3}\,xy + 4y^2 = 1$

a) What angle of rotation of axes will reduce the equation to standard form ?

$$\cot 2\alpha = \frac{2-4}{2\sqrt{3}} = \frac{-2}{2\sqrt{3}} = -\frac{1}{\sqrt{3}}$$



$$2\alpha = \frac{2\pi}{3}$$

$$\alpha = \frac{\pi}{3}$$

$$\sin\alpha = \frac{\sqrt{3}}{2} \qquad \cos\alpha = \cancel{\frac{\sqrt{3}}{2}}\ \frac{1}{2}$$

b) Expréss the new coordinates in terms of the old.

$$\bar{x} = \frac{1}{2}x + \frac{\sqrt{3}}{2}y$$

$$\bar{y} = -\frac{\sqrt{3}}{2}x + \frac{1}{2}y$$

8. The equation $x^2 + xy + y^2 = 1$ can be reduced to standard form by a rotation of axes of $\alpha = \frac{\pi}{4}$. What will be the equation in the new variables $\bar{x}$ and $\bar{y}$?

$$\cos\alpha = \sin\alpha = \frac{1}{\sqrt{2}}$$

$$x = \frac{1}{\sqrt{2}}\bar{x} - \frac{1}{\sqrt{2}}\bar{y} \qquad y = \frac{1}{\sqrt{2}}\bar{x} + \frac{1}{\sqrt{2}}\bar{y}$$

$$\frac{\bar{x}^2}{2} - \bar{x}\bar{y} + \frac{\bar{y}^2}{2} + \frac{1}{2}\bar{x}^2 - \frac{1}{2}\bar{y}^2 + \frac{1}{2}\bar{x}^2 + \bar{x}\bar{y} + \frac{1}{2}\bar{y}^2 = 1$$

$$\frac{3}{2}\bar{x}^2 + \frac{1}{2}\bar{y}^2 = 1$$

Write DNE for "Does not exist".

9. $\lim\limits_{x \downarrow 0} \frac{e^x}{x} = \frac{1}{\ln x} = \infty$

10. $\lim\limits_{x \to \infty} \frac{\ln x}{x^2} = \lim\limits_{x \to \infty} \frac{\frac{1}{x}}{2x} = \lim\limits_{x \to \infty} \frac{1}{2x^2} = 0$

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11. $\lim\limits_{x \to 0} \frac{\sin x}{x^2} = \lim\limits_{x \to 0} \frac{\cos x}{2x} = \infty$

12. $\int_2^\infty \frac{1}{x \ln x}\, dx =$ $\quad \ln \ln x \Big|_2^R = \ln \ln R - \ln \ln 2$

diverges

give full credit

13. $\int_0^1 \frac{1}{(x-1)^2}\, dx =$ $\quad \frac{(x-1)^{-1}}{-1} \Big|_0^t = -\frac{1}{t-1} + \frac{1}{-1}$

= diverges

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14. $\sum_{k=1}^{\infty} \frac{3}{2^k} = 3\sum_{k=0}^{\infty} \frac{\cancel{3}}{2^k} - 3$

$= \frac{3}{1-\frac{1}{2}} - 3 = 6 - 3 = 3$

Test for convergence. Specify tests used and show work.

15. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2+k}$ $\qquad \frac{1}{k^2+k} \le \frac{1}{k^2}$ con abs

comparison.

16. $\sum_{n=1}^{\infty} \frac{n+1}{n!}$ $\qquad \frac{n+2}{(n+1)!} \cdot \frac{n!}{n+1} = \frac{n+2}{(n+1)^2} \to 0$

con abs, ratio

test name?

17. $\sum_{n=2}^{\infty} \frac{n}{n+3}$ $\qquad \frac{n}{n+3} \to 1 \neq 0$ diverge

18. $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ $\qquad \sum \frac{1}{k^{1/2}}$ div

$\frac{1}{\sqrt{k}} \to 0$ dec

AST $\Rightarrow$ cond conv

19. Show that $\int_1^{\infty} \frac{x+1}{x^2-1}\,dx$ diverges.

$$\frac{x+1}{x^2-1} \geq \frac{x}{x^2} = \frac{1}{x}$$

div

20. For what values of x does the power series

$$\sum_{k=0}^{\infty} (-1)^k \frac{(k+1)(k+2)}{2^k} (x-3)^k$$

converge absolutely, conditionally, or diverge?

$$\frac{(k+2)(k+3)}{2^{k+1}} \frac{2^k}{(k+1)(k+2)} |x-3| = \frac{k+3}{k+1} \frac{|x-3|}{2} \longrightarrow \frac{|x-3|}{2}$$

abs con $\quad \frac{|x-3|}{2} < 1 \quad |x-3| < 2 \quad -2 < x-3 < 2$

$1 < x < 5$

div $\quad x \geq 5,\ x \leq 1$

$x = 1$

$\sum \frac{(-1)^k (k+1)(k+2)}{2^k} (-2)^k$ div

$x = 5$

$\sum \frac{(-1)^k (k+1)(k+2)}{2^k} 2^k$ div