MATH 232
FINAL EXAM
May 7, 1985

NAME KEY
( ) Same as test ( ) Double test
S-UP YES ( ) NO ( )

Time OK
Began leaving after 1.5 hrs
Lisa left at usual time
only a few left after 2 hrs
median = mean = 130

(30) 1. Let $\underset{\sim}{a} = \langle -1,2,5\rangle$ and $b = \langle 1,-5,6\rangle$.

Compute:

a) $\underset{\sim}{a} + \underset{\sim}{b}$ $\langle 0,-3,11\rangle$

b) $\underset{\sim}{a} - \underset{\sim}{b}$ $\langle -2,7,-1\rangle$

c) $3\underset{\sim}{a}$ $\langle -3,6,15\rangle$

d) $-\underset{\sim}{a}$ $\langle 1,-2,-5\rangle$

e) $\underset{\sim}{a} \cdot \underset{\sim}{b}$ $-1-10+30 = \boxed{19}$

f) $\underset{\sim}{a} \times \underset{\sim}{b}$ $\begin{vmatrix} i & j & k \\ -1 & 2 & 5 \\ 1 & -5 & 6 \end{vmatrix} = 12i + 5j + 5k - (2k - 25i - 6j)$

$= \cancel{10i + 30j + 11k} = \cancel{\langle 10, 30, 11\rangle}$

$37i + 11j + 3k = \boxed{\langle 37, 11, 3\rangle}$

g) $\|\underset{\sim}{a}\| = \sqrt{1+4+25}$
$\boxed{= \sqrt{30}}$

21

4 h) a unit vector (length one) in same direction as $\underset{\sim}{b}$

$$\frac{1}{\sqrt{1+25+36}}\langle 1,-5,6\rangle = \frac{1}{\sqrt{62}}\langle 1,-5,6\rangle = \left\langle \frac{1}{\sqrt{62}}, -\frac{5}{\sqrt{62}}, \frac{6}{\sqrt{62}}\right\rangle$$

5 i) the projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$.

$$\frac{\langle -1,2,5\rangle \cdot \langle 1,-5,6\rangle}{\|a\|^2}\, a$$

$$\frac{19}{30}\langle -1,2,5\rangle = \left\langle -\frac{19}{30}, \frac{38}{30}, \frac{95}{30}\right\rangle$$

30

(5) 2. Find the equation of the sphere with center (1,1,-2) and radius 9.

$(x-1)^2+(y-1)^2+(z+2)^2=81$

(5) 3. Find the equation of the plane which passes through the points (1,2,5), (2,1,3), and (7,8,-5).

$(2,1,3)-(1,2,5)=\langle 1,-1,-2\rangle$

$(7,8,-5)-(2,1,3)=\langle 5,7,-8\rangle$

$n=\begin{vmatrix} i & j & k \\ 1 & -1 & -2 \\ 5 & 7 & -8 \end{vmatrix} \begin{matrix} i & j \\ 1 & -1 \\ 5 & 7 \end{matrix} = 8i-10j+7k-(-5k-14i-8j)$

$=\langle 22,-2,12\rangle$

$22(x-1)-2(y-2)+12(z-5)=0$

$\boxed{22x-2y+12z=22-4+60=78}$

(15) 4. Consider the points A:(1,3,7), B:(8,-1,7), and C:(2,2,-8)

a) Write the vector which goes from A to B.

$\langle 7,-4,0\rangle$

b) Find the cosine of the angle between sides AB and AC.

$AC=\langle 1,-1,-15\rangle$

$\cos\theta=\frac{\langle 1,-1,-15\rangle\langle 7,-4,0\rangle}{\|\cdot\| \quad \|\cdot\|}=\frac{7+4}{\sqrt{227}\sqrt{65}}$

c) Is the triangle ABC a right triangle? why?

$AB\cdot AC\neq 0$

$AB\cdot BC=\langle 7,-4,0\rangle\langle -6,3,-15\rangle$

$=-42-12\neq 0$

$AC\cdot BC=\langle 1,-1,-15\rangle\langle -6,3,-15\rangle=-6-3+15^2\neq 0$

No!

25

(8) 5. Find the vector (or parametric) equation of the line of intersection of the planes $x + y + z = 3$ and $x - y + 2z = 2$. (Note: Verify that the point (1,1,1) lies on both planes).

$n_1 = \langle 1,1,1 \rangle$

$n_2 = \langle 1,-1,2 \rangle$

$$n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} \begin{matrix} i & j \\ 1 & 1 \\ 1 & -1 \end{matrix} = 2i + j - k$$

$-k + i + 2j$

$= \langle 3, -1, -2 \rangle$

$3 \cancel{(x-1) - (y-1) - 2(z-1) = 0}$

~~$3x - y - 2z = 3 - 1 - 2 = 0$~~

$\langle 1,1,1 \rangle + t\langle 3,-1,-2 \rangle = \langle 1+3t, 1-t, 1-2t \rangle$

(6) 6. Find the vector (or parametric) equation for the straight line which passes through the points (1,2,-5) and (1,0,7).

$\underline{v} = \langle 0, -2, 12 \rangle$

$\langle 1,2,-5 \rangle + t\langle 0,-2,12 \rangle$

$\langle 1, 2-2t, -5+12t \rangle$

(5) 7. Are the straight lines with parametric equations $x = 2t$, $y = 3t$, $z = -4t$ and $x = t$, $y = t$, $z = 2t$ orthogonal? Justify.

$2i + 3j - 4k$

$i + j + 2k$ ~~and~~.

$\cdot = 2 + 3 - 8 = -3$ ~~yes~~

NO

18

(25) 8. Consider the vector-valued function

$$\bar{F}(t) = (3t-t^3)\underline{i} + t^2\underline{j},\ 0 \le t \le 2.$$

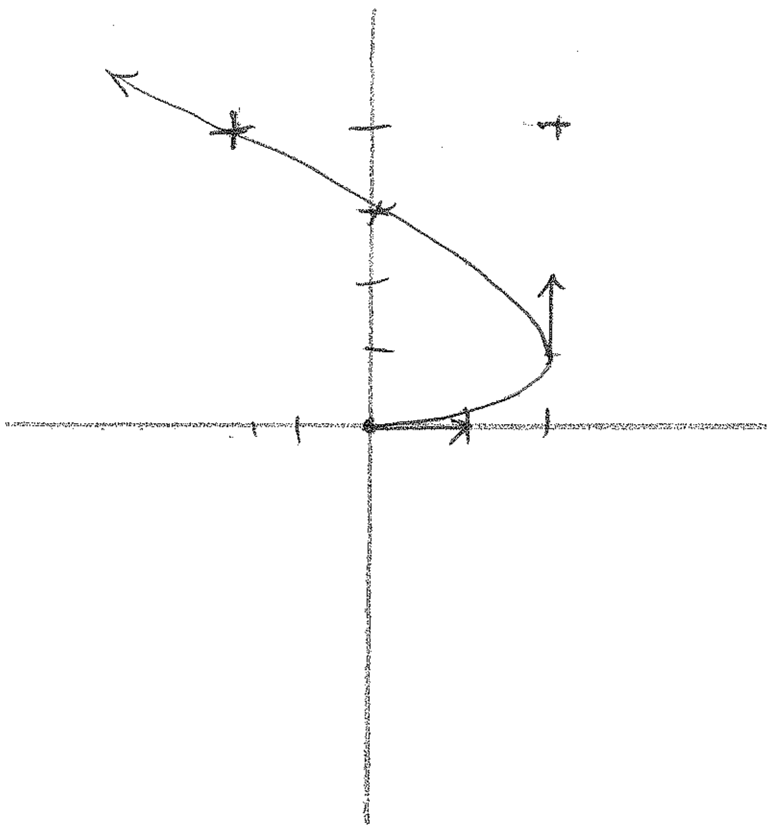
a) Where does the graph cross the axes?

b) Sketch the graph of $\bar{F}$.

c) Compute the unit tangent vector $\bar{T}$ for $t = 1$.

d) Compute the unit tangent vectors at the endpoints.

e) Draw the tangent vectors in parts c and d on the graph of part b.



a) $3t - t^3 = 0$

$t(3-t^2) = 0 \quad t = \pm\sqrt{3}$

$t = 0$

$\boxed{(0,0)}$

$\boxed{(0,3)}$

b) $t = 0 \quad (0,0)$

$t = 2 \quad (6-8, 4) = (-2, 4)$

$t = 1 \quad (2, 1)$

c) $\bar{F}'(t) = (3 - 3t^2)\bar{i} + 2t\bar{j}$

$t = 1$

$\bar{F}'(1) = 0\bar{i} + 2\bar{j}$

$\bar{T} = \bar{j}$

d. $t = 0$

$\bar{F}'(0) = 3\bar{i}$

$\bar{T} = \bar{i}$

$t = 2$

$\bar{F}'(2) = -9\bar{i} + 4\bar{j}$

$T = \frac{1}{\sqrt{97}}(-9i + 4j)$

e) tangents

25

(10) 9. Suppose position is given by the vector valued function

$$\underline{r}(t) = t^2\underline{i} + (t^3-1)\underline{j}$$.

a) Find the velocity and acceleration vectors.

b) Find the normal and tangential acceleration at $t = 1$.

a. $v(t) = r'(t) = 2t\,i + (3t^2)j$

$a(t) = 2i + 6t\,j$

b. $t=1$ $\quad v(1) = 2i + 3j$

$a(1) = 2i + 6j$

$$a_T = \frac{a\cdot v}{v\cdot v} v = \frac{\langle 2,6\rangle\langle 2,3\rangle}{13}\langle 2,3\rangle = \frac{22}{13}\langle 2,3\rangle$$

$$= \left\langle \frac{44}{13}, \frac{66}{13}\right\rangle$$

$$a_N = \langle 2,6\rangle - \frac{22}{13}\langle 2,3\rangle$$

$$= \left\langle -\frac{18}{13}, \frac{12}{13}\right\rangle$$

(10) 10. A bacteria culture growing under ideal conditions will grow at a rate proportional to the amount present. After two hours a count of 2000 grew to 4000. How long will it take for this culture to reach 5000?

$f'(t) = k f(t)$

$f(t) = Ce^{kt}$

$= 2000e^{kt}$

$4000 = 2000e^{k2}$

$2k = \ln 2$

$k = \frac{1}{2}\ln 2$

$f(t) = 2000e^{(\frac{1}{2}\ln 2)t}$

$5000 = 2000e^{\frac{1}{2}\ln 2 t}$

$\ln\frac{5}{2} = \frac{1}{2}\ln 2t$

$t = \frac{2\ln 5/2}{\ln 2} \approx 2.64$

(9) 11. Evaluate or simplify:

a) $D_x \arcsin \frac{x}{5} =$ $\frac{1}{\sqrt{1-(\frac{x}{5})^2}} \cdot \frac{1}{5} = \frac{1}{\sqrt{25-x^2}}$

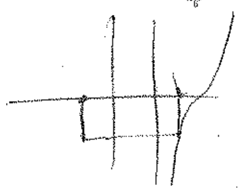
4

b) $\exp(2\ln x) =$ $e^{\ln x^2} = x^2$

2

c) $\arctan(\tan \frac{3\pi}{4}) =$ $-\frac{\pi}{4}$

2



(40) 12. Integrate the following:

a) $\int x \cos x \, dx =$ $x \sin x - \int \sin x \, dx$

$u = x \quad dv = \cos x \, dx$

$du = dx \quad v = \sin x$

$= x \sin x + \cos x + C$

16

b) $\int \frac{1}{\sqrt{4-x^2}} dx =$ $\int \frac{1}{\sqrt{1-(\frac{x^2}{4})}} \frac{1}{2} dx = \int \frac{1}{\sqrt{1-u^2}} du$

$u = \frac{x}{2}$

$du = \frac{1}{2} dx$

$= \arcsin u + C$

$= \arcsin \frac{x}{2} + C$

25

c) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{u}\,du = -\ln u + c$

$u = \cos x$ $\qquad = -\ln|\cos x| + c$

$du = -\sin x\,dx$

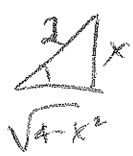
d) $\int x(\sec 2x^2)^2\,dx = \; = \int \frac{1}{4}(\sec u)^2\,du$

$u = 2x^2$

$du = 4x\,dx$ $\qquad = \frac{1}{4}\tan u + c$

$= \frac{1}{4}\tan 2x^2 + c$

e) $\int \frac{x^3}{\sqrt{4-x^2}}\,dx = \; = \int \frac{(2\sin u)^3}{\cancel{2\cos u}}\,\cancel{2\cos u}\,du = \int (2\sin u)^3\,du$



$\sin u = \frac{x}{2}$

$\cos u\,du = \frac{1}{2}\,dx$

$= 8\int (1-\cos^2 u)\sin u\,du$

$= -8\int (1-v^2)\,dv$

$= -8\left(v - \frac{v^3}{3}\right) + c$

$= -8\cos u + \frac{8}{3}(\cos u)^3 + c$

$= -4\sqrt{4-x^2} + \frac{1}{3}(4-x^2)^{3/2} + c$

or (2) $u = 4 - x^2$ $\quad x^2 = 4 - u$

$du = -2x\,dx$

$\int \frac{4-u}{\sqrt{u}} \cdot -\frac{1}{2}\,du$

$-\int 2u^{-1/2} - \frac{1}{2}u^{1/2}\,du$

$-\frac{2u^{1/2}}{\cancel{1/2}} + \frac{1}{2}\frac{u^{3/2}}{3/2} + c$

$-4u^{1/2} + \frac{1}{3}u^{3/2} + c$

$-4\sqrt{4-x^2} + \frac{1}{3}(4-x^2)^{3/2} + c$

(3) $u = x^3$ $\quad dv = \frac{x}{\sqrt{4-x^2}}\,dx$

$du = 3x^2\,dx$ $\quad v = \arcsin\frac{x}{2}$

(7) 13. Find the sum of the following series:

$$\sum_{n=1}^{\infty} (-1)^n \frac{2}{5^n} = 2\sum_{n=1}^{\infty}\left(\frac{-1}{5}\right)^{n}$$

$$= 2\left[\sum_{n=0}^{\infty}\left(-\frac{1}{5}\right)^n - 1\right]$$

$$= 2\left[\frac{1}{1+\frac{1}{5}} - 1\right] = 2\left[\frac{5}{6} - 1\right] = \frac{-2}{6} = \boxed{-\frac{1}{3}}$$

(6) 14. Use infinite series to write the repeating decimal .735735735... as a ratio of integers.

$$735 \sum_{n=1}^{\infty} (.001)^n = 735\left[\sum_{n=0}^{\infty}(.001)^n - 1\right]$$

$$= 735\left[\frac{1}{1-.001} - 1\right] = 735\left[\frac{1000}{999} - 1\right]$$

$$= 735\left[\frac{1}{999}\right] = \frac{735}{999}$$

(20) 15. Test each series for convergence (absolute or conditional) or divergence.

a) $\sum_{n=1}^{\infty} \frac{n^2+1}{n^2-n}$

$$\frac{n^2+1}{n^2-n} \not\to 0 \quad \text{diverges}$$

6

b) $\sum_{n=1}^{\infty} \frac{n}{n^2-4}$

$$\frac{n}{n^2-4} \geq \frac{n}{n^2} = \frac{1}{n} \qquad \sum \frac{1}{n} \text{ div}$$

$$\Rightarrow \sum \frac{n}{n^2-4} \text{ div} \quad CT$$

6

c) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}-1}$

$$\frac{1}{\sqrt{n}-1} \geq \frac{1}{\sqrt{n}} \qquad \sum \frac{1}{\sqrt{n}} \text{ div}$$

8

But $\frac{1}{\sqrt{n}-1}$ dec. $\to 0$

By AST $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}-1}$ conv (COND)