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Math 232
Final Exam
June 5, 1972

Name KEY

(20) 1. Evaluate the following limits:

a. $\lim_{n\to\infty} \frac{1+n^2}{2+n+n^3} = \lim_{n\to\infty} \frac{\frac{1}{n^3}+\frac{1}{n}}{\frac{2}{n^3}+\frac{1}{n^2}+1} = 0$

b. $\lim_{x\to\infty} \frac{x^2+1}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$

c. $\lim_{n\to\infty} \frac{\cos n\pi}{n\pi} = \quad \left|\frac{\cos n\pi}{n\pi}\right| \le \frac{1}{n\pi} \quad \lim_{n\to\infty} \frac{1}{n\pi} = 0 \Rightarrow = 0$

d. $\lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$

(15) 2. Evaluate, if possible:

a. $\lim_{t\to\infty}\int_0^\infty x\, e^{-x^2}\, dx = \lim_{t\to\infty} -\frac{1}{2}e^{-x^2}\Big|_0^t = \lim_{t\to\infty} -\frac{1}{2}e^{-t^2}+\frac{1}{2} = \frac{1}{2}$

b. $\int_{-1}^{2} \frac{1}{(x-2)^2}\, dx = \lim_{t\to 2^-} \int_{-1}^{t}(x-2)^{-2}\, dx = \lim_{t\to 2^-} -(x-2)^{-1}\Big|_{-1}^{t}$

$= \lim_{t\to 2^-} \frac{-1}{t-2} + \frac{1}{-3} \to \infty$

(10) 3. a. $\sum_{k=1}^{\infty} \frac{2^k+3^k}{7^k} = \sum_{k=0}^{\infty} \frac{2^k}{7^k} - 1 + \sum_{k=0}^{\infty} \frac{3^k}{7^k} - 1 = \frac{1}{1-\frac{2}{7}} + \frac{1}{1-\frac{3}{7}} - 2$

$= \frac{7}{5} + \frac{7}{4} - 2 = \frac{28+35}{20} - 2 = \frac{63}{20} - 2 = \frac{23}{20} - \cancel{\frac{9}{10}}$

b. $\sum_{k=1}^{\infty}\left(\frac{1}{10}\right)^k = .111111\ldots$; Write as a fraction.

$\sum_{k=0}^{\infty}\left(\frac{1}{10}\right)^k - 1 = \frac{1}{1-\frac{1}{10}} - 1 = \frac{10}{9} - 1 = \frac{1}{9}$

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(2-) 4. Test for convergence (absolute, conditional), give reasons:

a. $\sum_{k=1}^{\infty} \frac{k}{k^3+1}$

$\frac{k}{k^3+1} \le \frac{k}{k^3} = \frac{1}{k^2}$, $\sum \frac{1}{k^2}$ converges,

so $\sum \frac{k}{k^3+1}$ converges by comp test

b. $\sum_{k=1}^{\infty} \frac{2k}{k+1}$

$\lim_{k\to\infty} \frac{2k}{k+1} = \lim_{k\to\infty} \frac{2}{1+\frac{1}{k}} = 2 \neq 0$ diverges

c. $\sum_{k=1}^{\infty} \frac{(-1)^k(k+2)}{k!}$

$\left|\frac{(-1)^{k+1}(k+3)}{(k+1)!} \cdot \frac{k!}{(-1)^k(k+2)}\right| = \left|\frac{-1}{(k+1)} \cdot \frac{k+3}{k+2}\right|$

$\lim_{k\to\infty} \left|\frac{k+3}{(k+1)(k+2)}\right| = 0$ conv abs.

(20) 5. a. $\langle 5,7\rangle + 6\langle 2,1\rangle =$ $\langle 5,7\rangle + \langle 12,6\rangle = \langle 17,13\rangle$

b. $\langle 2,7\rangle \cdot \langle 6,3\rangle =$ $2\cdot 6 + 7\cdot 3 = 12+21 = 33$

c. Find the cosine of the angle between the vectors $\langle 2,1\rangle$ and $\langle 3,2\rangle$.

$\langle 2,1\rangle \cdot \langle 3,2\rangle = 6+2 = 8$

$|\langle 2,1\rangle| = \sqrt{4+1} = \sqrt{5}$ $\quad \cos\theta = \frac{8}{\sqrt{5}\sqrt{13}}$

$|\langle 3,2\rangle| = \sqrt{9+4} = \sqrt{13}$

d. Find a vector perpendicular to $\langle 3,7\rangle$.

$\langle 7,-3\rangle$ or $\langle -7,3\rangle$

75

(10) 6. Consider the curve given by the parametric equations

$$x = \cos^3 t \,, \quad y = \sin^3 t \,.$$

Find the arc length of thês curve in the first quadrant.

$x' = -3\cos^2 t \sin t \qquad y' = 3\sin^2 t \cos t$

$F'(t) = \langle -3\cos^2 t \sin t,\ 3\sin^2 t \cos t \rangle$

10

$|F'(t)| = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}$

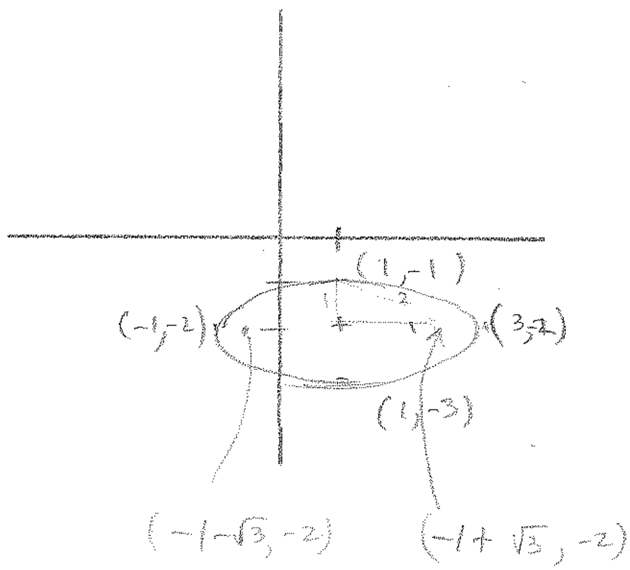
$= 3|\cos t||\sin t|\sqrt{\cos^2 + \sin^2} = 3|\cos t||\sin t|$

$AL = \int_0^{\pi/2} 3\cos t \sin t \, dt = -\frac{3}{2}\cos^2 t \Big|_0^{\pi/2} = -\frac{3}{2}[0-1] = \frac{3}{2}$

or $\frac{3}{2}\sin^2 t \Big|_0^{\pi/2} = \frac{3}{2} - 0$

(15) 7. Consider the equation $x^2 + 4y^2 - 2x + 16y = -13$.

a. Put this equation into standard form and carefully sketch its graph. Plot and give the coordinates of the vertices and foci.



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$x^2 - 2x + 1 + 4y^2 + 16y + 16 = -13 + 1 + 16$

$(x-1)^2 + 4(y+2)^2 = 4$

$\frac{(x-1)^2}{4} + (y+2)^2 = 1$

ellipse (1,-2)

$\sqrt{4-1} = \sqrt{3}$

b. Convert this equation into polar form.

4

$x = r\cos\theta \qquad y = r\sin\theta$

$r^2\cos^2\theta + 4r^2\sin^2\theta - 2r\cos\theta + 16r\sin\theta = -13$

25

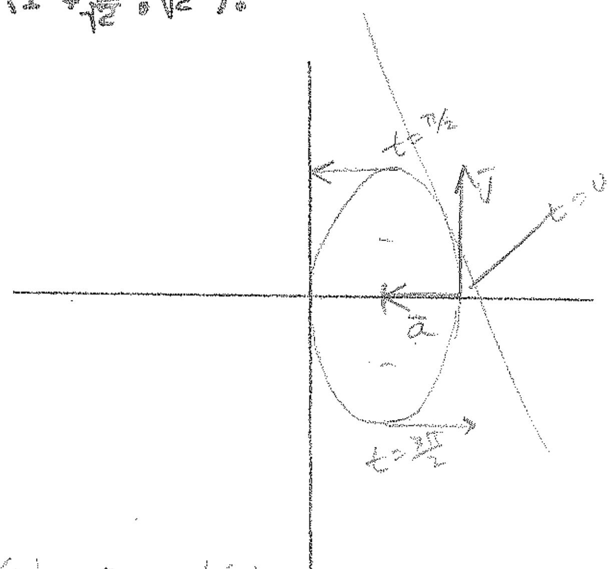
(35) 8. A particle is travelling along a path, and its position is given by $P(t) = (1 + \cos t,\ 2\sin t)$, where t is time in seconds, $t \geq 0$.

8-10 a. Sketch the path.

-1 8 b. Find the coordinates of the point(s) at which the velocity of the particle is horizontal.

7 c. Find the velocity and acceleration at time $t=0$.

10 d. Find the equation of the line tangent to the path at the point $(1 + \frac{1}{\sqrt{2}}, \sqrt{2})$.



a. $x = 1 + \cos t \quad y = 2\sin t$

$x - 1 = \cos t \quad \frac{y}{2} = \sin t$

$(x-1)^2 + \left(\frac{y}{2}\right)^2 = 1$

ellipse
center $(1, 0)$
x axis 1
y axis 2

b. $x' = -\sin t \quad y' = 2\cos t$

$y' = 0 \quad \cos t = 0$

$t = \frac{\pi}{2}, \frac{3\pi}{2}$

$t = \frac{\pi}{2} \quad x = 1 \quad y = 2$

$x' = -1$

$t = \frac{3\pi}{2} \quad x = 1, \quad y = -2$

$x' = 1$

c. $x'(0) = 0 \quad y'(0) = 2 \quad v = \langle 0, 2 \rangle$

$x''(t) = -\cos t \quad y''(t) = -2\sin t$

$x''(0) = -1 \quad y''(0) = 0$

$\vec{a} = \langle -1, 0 \rangle$

d. $t = \frac{\pi}{4}$

$x'\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} \quad y'\left(\frac{\pi}{4}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$

$\vec{m} = \langle -\frac{1}{\sqrt{2}}, \sqrt{2} \rangle$

$m = \frac{\sqrt{2}}{-1/\sqrt{2}} = -2$

$\frac{y - \sqrt{2}}{x - 1 - \frac{1}{\sqrt{2}}} = -2 \quad y - \sqrt{2} = -2x + 2 + \sqrt{2}$

$y = -2x + 2 + 2\sqrt{2}$

35

Problems 9 through 12 will be graded on the basis of 5 points. Then the highest score will be multiplied by 4, the next by 3, the next by 2, and the last by 1. Thus you can omit one problem and still get 45, omit 2 and still get 35. Total possible 50 points.

9. Prove that $\lim\limits_{n \to \infty} (a_n + b_n) = \lim\limits_{n \to \infty} a_n + \lim\limits_{n \to \infty} b_n$.

<u>OR</u> prove that $\lim\limits_{n \to \infty} r a_n = r \lim\limits_{n \to \infty} a_n$.

10. Carefully sketch the graph of the polar equation $r = 2 \sin 3\theta$. Plot all points with $3\theta = 0, \pi/6$, etc.

a. For what values does the curve pass through the pole?

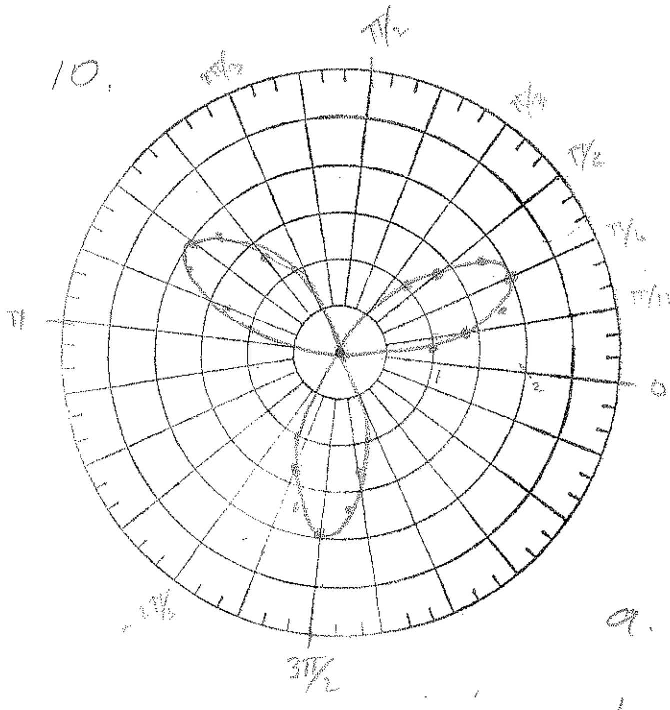
b. Find the area of the region enclosed by the graph (set up).

11. Find the interval of convergence of the power series

$$\sum_{k=1}^{\infty} \frac{(-2)^k}{k+1}(x+1)^k$$

12. Find the centroid of the parabolic region bounded by $y = 3x^2 - 6x$ and the x-axis.

10.



θ	3θ	$2 \sin 3\theta$
0	0	0
$\frac{\pi}{18}$	$\pi/6$	$\frac{1}{2} \cdot 2 = 1$
$\pi/12$	$\pi/4$	$2 \cdot \frac{1}{\sqrt{2}} = \sqrt{2}$
$\pi/9$	$\pi/3$	$2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$
$\pi/6$	$\pi/2$	2
$\frac{2\pi}{9}$	$\frac{2\pi}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\sqrt{2}$
$\frac{5\pi}{18}$	$\frac{5\pi}{6}$	1
$\pi/3$	π	0

a. $\theta = \pi/3, 2\pi/3, \pi$

b. $\frac{1}{2}\int_0^{\pi} 4 \sin^2 3\theta \, d\theta$

or $\frac{3}{2}\int_0^{\pi/3} 4 \sin^2 3\theta \, d\theta$

50

11. $\sum \frac{(-2)^k}{k+1}(x+1)^k$

$$\left| \frac{(-2)^{k+1}}{k+2}(x+1)^{k+1} \cdot \frac{k+1}{(x+1)^k(-2)^k} \right| = \left| (-2)(x+1)\frac{k+1}{k+2} \right|$$

$$\lim_{k\to\infty} \left| (-2)(x+1)\frac{k+1}{k+2} \right| = 2|x+1| < 1$$

$$|x+1| < \frac{1}{2} \qquad -\frac{1}{2} < x+1 < \frac{1}{2}$$

$$-\frac{3}{2} < x < -\frac{1}{2}$$

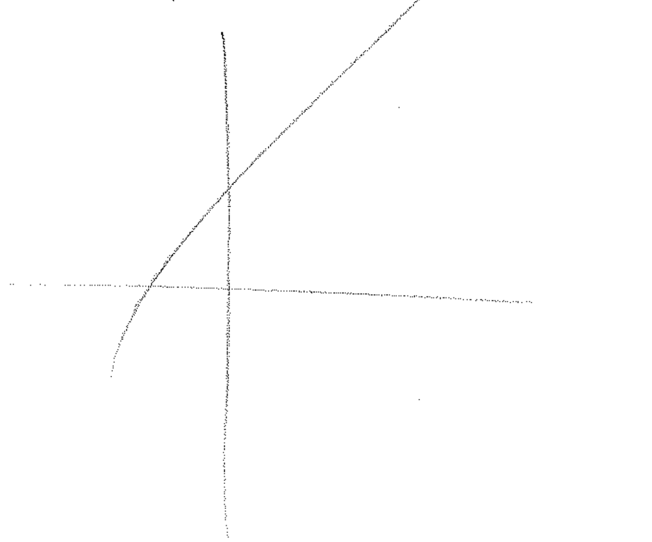
$x = -\frac{3}{2}$ $\quad \sum \frac{(-2)^k}{k+1}\left(-\frac{1}{2}\right)^k = \sum \frac{1}{k+1}$ diverges

$x = -\frac{1}{2}$ $\quad \sum \frac{(-2)^k}{k+1}\cdot\left(\frac{1}{2}\right)^k = \sum \frac{(-1)^k}{k+1}$ converges by AST

Interval of convergence $\left(-\frac{3}{2}, -\frac{1}{2}\right]$.

12. $y = 3x^2 - 6x$, $y = 0$

$y + 3 = 3(x^2 - 2x + 1)$

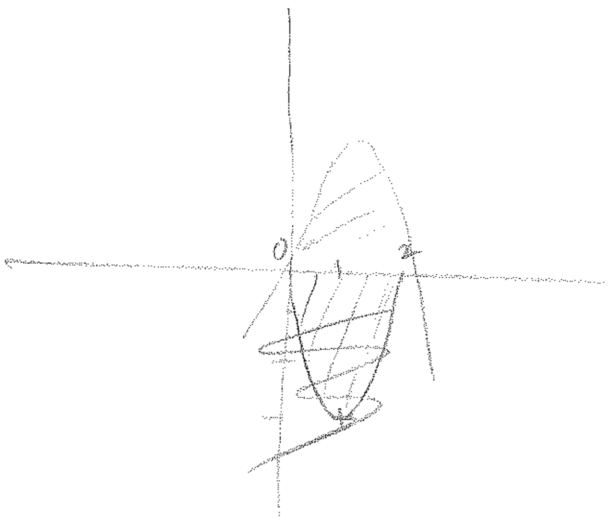


$-3x^2+6x$

12. $y = \cancel{3x^2 - 6x}$ $\qquad y = 0$

$y + 3 = -3x^2 + 6x + 3$

$y + 3 = -3(x-1)^2$



$\text{area} = -\int_0^2 3x^2 - 6x\,dx$

$= -(x^3 - 3x^2)\Big|_0^2$

$= -(8 - 12) = 4$

$M_x = \int_0^2 \frac{f(x)^2}{2}\,dx$

$= \frac{1}{2}\int_0^2 9x^4 - 36x^3 + 36x^2\,dx$

$= \frac{1}{2}\left[\frac{9x^5}{5} - 9x^4 + 12x^3\right]_0^2$

$= \frac{1}{2}\left[\frac{9}{5}\cdot 32 - 9\cdot 16 + 12\cdot 8\right]$

$M_y = \int_0^2 x f(x)\,dx$

$\int_0^2 x(-3x^2+6x)\,dx = \int_0^2 -3x^3 + 6x^2\,dx = -\frac{3x^4}{4} + 2x^3\Big|_0^2$

$= -3\cdot 4 + 16 = 4$

$\bar{X} = \frac{4}{4} = 1$

$M_x = \frac{9}{5}\cdot 16 - 9\cdot 8 + 12\cdot 4$ $\quad -72 + 48$

$= \frac{9}{5}\cdot 16 - 24$ $\qquad \frac{16}{9}$ $\quad 144$

$= \frac{144 - 120}{5}$

$= \frac{24}{5}$

$\bar{Y} = \frac{\frac{24}{5}}{4} = \frac{6}{5}$

$\left(1, \frac{6}{5}\right)$