Math 232
Test II
May 15, 1972

Name *KEY*

(10) 1. Complete the following definitions:

a. An infinite series converges if and only if *the sequence of partial sums $S_n = \sum_{k=1}^{n} a_k$ has a finite limit*

b. An infinite series $\sum a_n$ converges absolutely if and only if *the series $\sum |a_n|$ converges*

(10) 2. Write out (do not add) the fifth partial sum of each of the following infinite series:

a. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(k+2)}{(k+1)^2}$, $\quad \frac{3}{2^2} - \frac{4}{3^2} + \frac{5}{4^2} - \frac{6}{5^2} + \frac{7}{6^2}$

b. $\sum_{k=1}^{\infty} \frac{k}{k+1} x^k$, $\quad \frac{1}{2}x + \frac{2}{3}x^2 + \frac{3}{4}x^3 + \frac{4}{5}x^4 + \frac{5}{6}x^5$

c. $\sum_{k=1}^{\infty} \frac{2^{k+1}-1}{k+3}$, $\quad \frac{2^2-1}{4} + \frac{2^3-1}{5} + \frac{2^4-1}{6} + \frac{2^5-1}{7} + \frac{2^6-1}{8}$

(10) 3. Show that the infinite series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges.

$$\int_1^{\infty} \frac{1}{x^2}dx = \lim_{L\to\infty} \int_1^{L} x^{-2}dx = \lim_{L\to\infty} -x^{-1}\Big|_1^L$$

$$= \lim_{L\to\infty} -\frac{1}{L} + 1 = 1 < \infty$$

So $\sum \frac{1}{k^2}$ converges by the integral test.

30

7. Test each of the following infinite series for convergence. Tell whether the convergence is absolute when the series has negative terms.

a. $\sum_{k=1}^{\infty} \frac{1-k}{2+k}$

$\lim_{k\to\infty} \frac{1-k}{2+k} = -1 \neq 0$ diverges

b. $\sum_{k=1}^{\infty} \frac{2k}{k^4+1}$

$\frac{2k}{k^4+1} \leq \frac{2k}{k^4} = \frac{2}{k^3}$

$\sum \frac{2}{k^3}$ converges $(3>1)$ so $\sum \frac{2k}{k^4+1}$ converges by comparison test

c. $\sum_{k=1}^{\infty} \frac{(-1)^k}{2k+3}$

$\frac{1}{2k+3} = \underbrace{\frac{k}{2k+3}}_{c_k} \frac{1}{k}$, $\lim_{k\to\infty} \frac{k}{2k+3} = \frac{1}{2}$,

so since $\sum \frac{1}{k}$ div, $\sum \frac{k}{2k+3}$ div by limit comp test.

But $\frac{1}{2k+3}$ dec, nonneg, $\lim \frac{1}{2k+3} = 0$, so by AST, $\sum \frac{(-1)^k}{2k+3}$ conv. So converges conditionally.

d. $\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+3)}$

$\frac{1}{(k+1)(k+3)} \leq \frac{1}{k^2}$

$\sum \frac{1}{k^2}$ con, so $\sum \frac{1}{(k+1)(k+3)}$ conv by comp test

e. $\sum_{k=1}^{\infty} \frac{2^k}{(k+1)!}$ $\lim_{k\to\infty} \left| \frac{2^{k+1}}{(k+2)!} \cdot \frac{(k+1)!}{2^k} \right| = \lim_{k\to\infty} \left| \frac{2}{k+2} \right| = 0 < 1$

so conv abs by ratio test.

40

(15) 5. Find the sum of each of the following series:

a. $\sum_{k=1}^{\infty} \frac{2\cdot 2^k}{5^{k+1}}$ $= \frac{2}{5}\sum_{k=1}^{\infty}\left(\frac{2}{5}\right)^k = \frac{2}{5}\left[\sum_{k=0}^{\infty}\left(\frac{2}{5}\right)^k - 1\right] = \frac{2}{5}\left[\frac{1}{1-\frac{2}{5}} - 1\right]$

$= \frac{2}{5}\left[\frac{5}{3} - 1\right] = \frac{2}{5}\left[\frac{2}{3}\right] = \frac{4}{15}$

b. $\sum_{k=1}^{\infty} \frac{3^k+1}{5^k}$ $= \sum_{k=1}^{\infty}\left(\frac{3}{5}\right)^k + \sum_{k=1}^{\infty}\left(\frac{1}{5}\right)^k = \frac{1}{1-\frac{3}{5}} - 1 + \frac{1}{1-\frac{1}{5}} - 1$

$= \frac{5}{2} - 1 + \frac{5}{4} - 1 = \frac{15}{4} - 2 = \frac{7}{4}$

(15) 6. Find the interval of convergence of the power series:

$$\sum_{k=1}^{\infty} \frac{(k+1)^2}{2^k}(x-1)^k$$

$\left|\frac{(k+2)^2(x-1)^{k+1}}{2^{k+1}} \cdot \frac{2^k}{(k+1)^2(x-1)^k}\right| = \frac{|x-1|}{2}\cdot\frac{(k+2)^2}{(k+1)^2}$

Ratio test

$\lim_{k\to\infty} \frac{|x-1|}{2}\frac{(k+2)^2}{(k+1)^2} = \frac{|x-1|}{2}$

abs conv $\quad |x-1| < 2 \quad -2 < x-1 < 2 \quad -1 < x < 3$

div $\quad |x-1| > 2 \quad x > 3$ or $x < -1$

$x = 3 \quad \sum_{k=1}^{\infty}\frac{(k+1)^2}{2^k}2^k \quad (k+1)^2 \not\to 0 \quad$ div

$x = -1 \quad \sum_{k=1}^{\infty}\frac{(k+1)^2(-1)^k}{2^k}(2)^k \quad \not\to 0 \quad$ div

$(-1, 3)$

30