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MATH 233

Final Exam

December 17, 1985

VBM

Name 170

Double ☐ equal ☐ weight to a test

~~med 80%~~
med 78%

Show work. Give solutions, not just answers.

1. Compute ∇f for $f(x,y,z) = x^2y^3z - xy\cos z$. (5)

$$\nabla f = \langle 2xy^3z - y\cos z,\ x^2 3y^2 z - x\cos z,\ x^2y^3 + xy\sin z \rangle$$

2. Compute $\int_C x^2 + y\, ds$, where C is the straight line starting at (1,1) and ending at (3,5). (10)

$$\langle x,y\rangle = \langle 1,1\rangle + t\langle 2,4\rangle = \langle 1+2t,\ 1+4t\rangle$$

$$r'(t) = \langle 2,4\rangle \quad \|r'\| = \sqrt{20} = 2\sqrt5$$

$$\int_0^1 \left((1+2t)^2 + (1+4t)\right) 2\sqrt5\, dt = 2\sqrt5\int_0^1 1+4t+4t^2+1+4t\, dt$$

$$= 2\sqrt5\int_0^1 2+8t+4t^2\, dt = 2\sqrt5\left(2t+4t^2+\frac{4t^3}{3}\right)\Big|_0^1 = 2\sqrt5\left(6+\frac{4}{3}\right) = \frac{44\sqrt5}{3}$$

$$= 32.7956$$

3. Is $(3x^2 + y^2)\,dx + (2xy + x^3 + \cos x)\,dy$ an exact differential? Justify. (5)

$$\frac{\partial}{\partial y}(3x^2+y^2) = 2y \qquad \frac{\partial}{\partial x}(2xy+x^3+\cos x) = 2y + x^2 - \sin x$$

no!

4. Use differentials to approximate (.98) sin(.01). (5)

Bad choice

$$f(x,y) = x\sin y \qquad dx = -.02 \quad dy = .01 \quad x=1,\ y=0$$

$$dz = (\sin y)\,dx + (x\cos y)\,dy$$

$$dz = 0 + dy = .01$$

$$f(1,0) = 0 \qquad \boxed{.01}$$

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5. Find general solutions for each of these differential equations: (30)

a. $y' = x^2 / y$

$$y y' = x^2 \qquad \int y\,dy = \int x^2\,dx$$

$$\frac{y^2}{2} = \frac{x^3}{3} + C$$

b. $y' = (2x - 2y)/(x+y)$ typo

$$u = \frac{y}{x} \qquad ux = y$$

$$u'x + u = y'$$

$$y' = \frac{2-2u}{1+u}$$

$$\frac{2-2u}{1+u} = u'x + u$$

$$u'x = \frac{2-2u}{1+u} - \frac{u(1+u)}{1+u}$$

$$= \frac{2-3u-u^2}{1+u}$$

$$\frac{1}{x} = \frac{1+u}{2-3u-u^2} u'$$

$$-3u - 2u$$

$$\ln x = \int \frac{1+u}{2-3u-u^2}\,du$$

$\cancel{u + 3u^2}$

c. $y' = (x^2 - y) / (x-y)$

$$x - y\, y' = x^2 - y$$

$$x^2 - y\,dx - (x-y)\,dy = 0$$

$$\frac{\partial}{\partial y}(x^2 - y) = -1$$

$$\frac{\partial}{\partial x}(-x+y) = -1$$

$$\int_0^x x^2 - y\,dx + \int_0^y y\,dy = C$$

$$\boxed{\frac{x^3}{3} - xy + \frac{y^2}{2} = C}$$

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(30)

6. Let $f(x,y) = \begin{cases} \cos y, & -\pi/2 < y < \pi/2 \\ 0, & \text{elsewhere.} \end{cases}$

a. Compute: $f(0,0)$, $f(\pi,0)$, $f(0,\pi)$.

b. For what points is this function discontinuous?

c. Sketch the graph, and give its name or type if it has one.

d. Find the equation of the plane which is tangent to this surface for $(x,y) = (\pi/2, \pi/4)$.

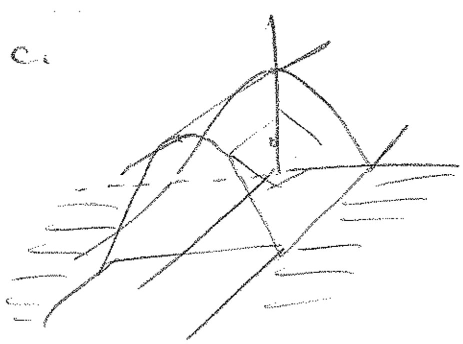
a. $f(0,0) = \cos 0 = 1$

$f(\pi,0) = \cos 0 = 1$

$f(0,\pi) = 0$

b. none

c.



cylinder

d. $F(x,y,z) = \cos y - z \qquad z = \cos y$

$\nabla F = \langle 0, -\sin y, -1 \rangle$

at $(\pi/2, \pi/4)$ $\quad \nabla F = \langle 0, -1/\sqrt{2}, -1 \rangle$

$F\left(\frac{\pi}{2}, \frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$-1/\sqrt{2}\left(y - \frac{\pi}{4}\right) - \left(z - 1/\sqrt{2}\right) = 0$

$-\frac{\sqrt{2}}{2} y - z + \left(\frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = 0$

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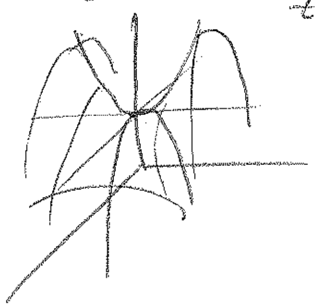
7. Consider the surface defined by $x^2 - y^2 + z = 1$. (20)

a. Sketch this surface. Give its name or type if it has one.

b. Find the point(s) on the surface for which the tangent plane is parallel to the plane $2x - 4y - z = 5$.

$z = y^2 - x^2 + 1$

a



hyperbolic paraboloid

$(-1,$

b. $f(x,y,z) = x^2 - y^2 + z - 1$

$\nabla f = \langle 2x, -2y, 1 \rangle$

$\underline{n} = \langle 2, -4, -1 \rangle$

$\nabla f = \lambda \underline{n}$

$\langle 2x, -2y, 1 \rangle = \lambda \langle 2, -4, -1 \rangle$

$\lambda = -1$

$2x = 2\lambda \qquad -2y = -4\lambda$

$2x = -2 \quad x = -1 \qquad y = 2\lambda = -2$

$z = \cancel{(-1)^2 - (-2)^2 + 1}$
$= (-2)^2 - (-1)^2 + 1$
$= 4$

$\boxed{(-1, -2, 4)}$

8. Suppose that a radioactive isotope has a half life of 30 seconds. If we start with 10 grams, how long will it take for the amount present to reach 3 grams? (10)

$f(t) = Ce^{kt}$
$= 10e^{kt}$

$5 = 10e^{k30}$

$\frac{1}{2} = e^{k30}$

$k30 = \ln\frac{1}{2}$

$k = \frac{1}{30}\ln(\frac{1}{2})$

$f(t) = 10e^{\frac{1}{30}(\ln\frac{1}{2})t}$

$\cancel{f(3) = 10e^{\frac{3}{30}\ln(\frac{1}{2})}}$

$= \cancel{10(\frac{1}{2})^{1/10}}$

$3 = 10e^{\frac{1}{30}\ln(\frac{1}{2})t}$

$\ln\left(\frac{3}{10}\right) = \frac{1}{30}\ln(\frac{1}{2})t$

$t = \frac{30\ln(.3)}{\ln(.5)}$

$= 52.1$

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9. Find the relative maxima and minima for f(x,y) = xy − sin x . (10)

$f_x = y - \cos x = 0 \qquad x = 0$

$f_y = x = 0 \qquad y - 1 = 0 \;\; y = 1 \qquad (0,1)$

$f_{xx} = +\sin x$

$f_{yy} = 0$

$f_{xy} = 1$

$M = 0 - 1^2 = -1 < 0$ saddle

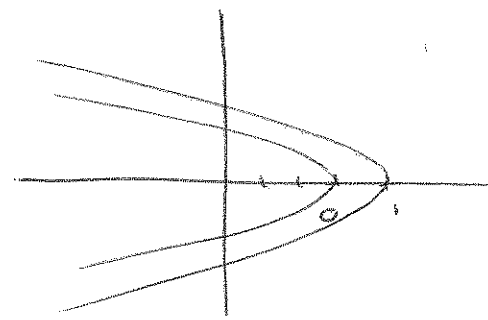
none

10. Draw two different level curves for f(x,y) = y² + 2x − 3 . Label. (10)

$y^2 + 2x - 3 = 0$

$2x = -y^2 + 3$

$2x = -y^2 + 3 + 1$



11. Let z = x²y + y³ . At the point (2,3) x is increasing at the rate of 3 in./sec. and y is decreasing at the rate of 2 in./sec. At what rate is z changing? Is z increasing or decreasing? (10)

$$\frac{dz}{dt} = 2xy \underbrace{\frac{dx}{dt}}_{=3} + (x^2 + 3y^2) \underbrace{\frac{dy}{dt}}_{=-2}$$

at (2,3)

$$\frac{dz}{dt} = 12(3) + 31(-2) = 36 - 62 = -26 \text{ in/sec}$$

dec

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12. Compute the directional derivative of $f(x,y,z) = x^2 - xz^3 + 2y$ at the point $(1,2,4)$ in the direction from there towards the point $(2,3,2)$. (5)

$\nabla f = \langle 2x - z^3, 2, -3xz^2 \rangle$

at $(1,2,4)$ $\nabla f = \langle 2-64, 2, -48 \rangle = \langle -62, 2, -48 \rangle$ $\frac{16}{6}$

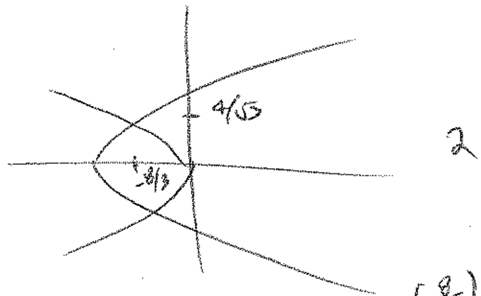
$u \quad \frac{\langle 1, 1, -2 \rangle}{\sqrt{6}}$

$D_u = \langle -62, 2, -48 \rangle \cdot \frac{\langle 1,1,-2 \rangle}{\sqrt{6}} = +\frac{\cancel{252}\ 156}{\sqrt{6}} = +6\sqrt{6}$

-62
$+96$
$\cancel{252}$
$\cancel{156}$

13. Use ~~double integrals~~ to find the area of the region bounded by the curves $2x = -y^2$ and $x-8 = y^2$. (10)

$x + 8 = y^2$



$2 \int_0^{4/\sqrt{3}} \int_{y^2-8}^{-y^2/2} 1 \, dx \, dy$

$x + 8 = -2x$

$3x = -8$

$x = -\frac{8}{3}$

$y^2 = x + 8 = 8 - 8/3$

$= \frac{24-8}{3} = \frac{16}{3}, \quad y = 4/\sqrt{3}$

$y^2 = -2(-\frac{8}{3}) = \frac{16}{3}$

$2 \int_{-4/\sqrt{3}}^{4/\sqrt{3}} -y^2/2 - (y^2 - 8) \, dx$

$= 2 \int_{-4/\sqrt{3}}^{4/\sqrt{3}} -\frac{3y^2}{2} + 8 \, dx$

$= 2\left(-\frac{y^3}{2} + 8x \right) \Big|_{-4/\sqrt{3}}^{4/\sqrt{3}}$

$= 2\left(-\left(\frac{4}{\sqrt{3}}\right)^3 \frac{1}{2} + 8 \cdot \frac{4}{\sqrt{3}} \right)$

$= \frac{-64}{3\sqrt{3}} + \frac{64}{\sqrt{3}}$

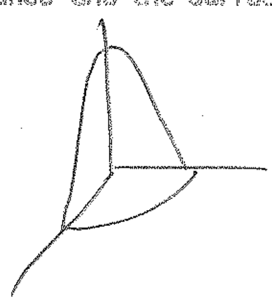
$= \frac{128}{3\sqrt{3}}$

$\frac{64}{\sqrt{3}} \left(\frac{2}{3}\right)$

$\frac{128}{3\sqrt{3}}$

$= \frac{128\sqrt{3}}{9}$

14. Find the volume of the region in the first octant which is bounded by the coordinate planes and the surface $z = 4 - x^2 - y^2$. (10)



$\iiint dz \, dx \, dy$

$= \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} dz \, r \, dr \, d\theta$

$= \int_0^{2\pi/2} \int_0^2 4r - r^3 \, dr \, d\theta$

$= \int_0^{2\pi/2} \frac{4r^2}{2} - \frac{r^4}{4} \Big|_0^2 d\theta = \int_0^{2\pi/2} 8 - 4 \, d\theta = 4\left(\frac{2\pi}{2}\right)$

$= 2\pi$

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