MATH 233 TEST II NAME
November 1, 1985

Maybe key (2 pubs?)
med 75
Most 60-92

Show work to justify your answers. Use methods covered in class since last test.

(35) 1. Compute the following:

a. Directional derivatives of $f(x,y) = x^3 - 7xy$ at $(1,2)$ in the direction $i + 2j$

$$\underline{u} = \frac{1}{\sqrt{5}}(1,2)$$

$$\nabla f = \langle 3x^2 - 7y, -7x \rangle$$

at $(1,2)$ $\langle 3-14, -7 \rangle = \langle -11, -7 \rangle$

$$D_u f = \langle -11, -7 \rangle \langle 1, 2 \rangle \frac{1}{\sqrt{5}} = \frac{-11-14}{\sqrt{5}} = -\frac{25}{\sqrt{5}} = -5\sqrt{5}$$

b. Gradient of $f(x,y) = x \sin y + y^2$

$$\nabla f = \langle \sin y, x \cos y + 2y \rangle$$

c. Normal vector to the surface $z = x^2 + 3xy^2$ at the point $(1,1,4)$.

$$G(x,y,z) = x^2 + 3xy^2 - z \qquad \text{at } (1,1,4)$$

$$\nabla G = \langle 2x + 3y^2, 6xy, -1 \rangle \qquad \nabla G = \underline{u} = \langle 5, 6, -1 \rangle$$

d. Critical points of the function $f(x,y) = x^2 - xy + 2$

$$\frac{\partial f}{\partial x} = 2x - y = 0$$

$$\frac{\partial f}{\partial y} = -x = 0 \qquad x = 0, \quad y = 0 \qquad (0,0)$$

e. $\int_1^2 \int_3^4 x^2 + y \, dx \, dy = \int_1^2 \frac{x^3}{3} + yx \Big|_3^4 dy = \int_1^2 \frac{64}{3} + 4y - 9 - 3y \, dy$

$$= \int_1^2 \frac{37}{3} + y \, dy = \frac{37}{3}y + \frac{y^2}{2} \Big|_1^2 = \frac{74}{3} + \frac{4}{2} - \frac{37}{3} - \frac{1}{2}$$

$$= \frac{37}{3} + 2 - \frac{1}{2} = \frac{37}{3} + \frac{3}{2} = \frac{83}{6}$$

$$\begin{array}{r} 64 \\ -27 \\ \hline 37 \end{array}$$

$$13\tfrac{5}{6} \qquad \frac{74}{6} + \frac{9}{6} = \frac{83}{6}$$

$$6 \overline{)83} = 13, \ \text{remainder } 5$$

-2 if get to here

f. $\int_0^1 \int_1^{x+1} \sqrt{1-x^2}\, dy\, dx$ $\qquad \int_0^1 \sqrt{1-x^2}\,(x+1-1)\, dx = \int_0^1 \sqrt{1-x^2}\, x\, dx$

$= -\frac{(1-x^2)^{3/2}}{3/2}\frac{1}{2}\Big|_0^1 = 0 - \left(-\frac{2}{3}\cdot\frac{1}{2}\right) = \frac{1}{3}$

g. $\int_0^\pi \int_0^\theta r^3\, dr\, d\theta$ $\qquad \int_0^\pi \frac{r^4}{4}\Big|_0^\theta d\theta = \int_0^\pi \frac{\theta^4}{4} - 0\, d\theta = \frac{\theta^5}{20}\Big|_0^\pi = \frac{\pi^5}{20}$

(7) 2. Find the direction in which $f(x,y) = x^3 + 2y^2$ increases most rapidly at the point $(1,2)$. What is this rate of increase?

$\nabla f = \langle 3x^2, 4y \rangle$

at $(1,2)$ $\qquad \nabla f = \langle 3, 8 \rangle$

rate $= \|\langle 3, 8\rangle\| = \sqrt{64+9} = \sqrt{73}$

2

(10) 3. Find the equation of the plane tangent to the surface which is the graph of $f(x,y) = xy - \sin x$, at the point $(0, \frac{\pi}{2})$.

$\nabla \cancel{f} = \langle y - \cos x,\ x, \cancel{z} - 1 \rangle$

$\underline{n} = \nabla \cancel{f}(0, \frac{\pi}{2}, ?) = \langle \frac{\pi}{2} - \cos 0, 0, -1 \rangle$

$= \langle \frac{\pi}{2} - 1, 0, -1 \rangle$

2d

-7

$f(0, \frac{\pi}{2}) = 0 - \sin 0 = 0$

$\left(\frac{\pi}{2} - 1\right)x - z = 0$

no plane

-10

(7) 4. Find the point(s) on the surface $x^2 + y^2 - z^2 = 1$ at which the tangent is parallel to the xz plane.

$$\nabla f = \langle 2x, 2y, -2z \rangle = \lambda \langle 0, 1, 0 \rangle$$

1 of 4 handle

$x = 0$

$z = 0$

$2y = \lambda$

$2y = \lambda$

$-2z = 0$

$x^2 + y^2 = 1 \quad y = \pm 1$

$(0, \pm 1, 0)$

all missed same pts

(10) 5. The points (0,0) and (1,1) are critical points of the function $f(x,y) = x^3 + y^3 - 3xy$. Are these relative maxima, minima, or what?

$f_x = 3x^2 - 3y \qquad f_y = 3y^2 - 3x$

$f_{xx} = 6x \qquad f_{yy} = 6y \qquad f_{xy} = -3$

$f_{xx}(0,0) = 0 \qquad f_{yy}(0) = 0 \qquad M = -(-3)^2 < 0$ $\boxed{\text{saddle}}$

$f_{xx}(1,1) = 6 \qquad f_{yy}(1,1) = 6 \qquad M = 36 - (-3)^2 > 0$

$\boxed{\text{rel min}}$

-1 if no f_{xx}

MOST GOT 9/10

(10) 6. Find 3 positive numbers x, y, z which add up to 12 for which x^2yz is a maximum. (Notice: For this to be a maximum, none of $x, y,$ or z will be 0).

Dozen or so got to end.

$x^2y(12-x-y)$

$= 12x^2y - x^3y$

$- x^2y^2$

$f_x = 12x^2$

OK

about same diff

2013, did this

$x + y + z = 12$

x^2yz max

$\nabla f = \langle 2xyz, x^2z, x^2y \rangle \qquad \nabla g = \langle 1, 1, 1 \rangle$

$2xyz = \lambda \qquad x^2z = x^2y \qquad 2xy^2 = x^2y$

$x^2z = \lambda \qquad z = y \qquad 2y = x$

$x^2y = \lambda$

$2y + y + y = 12 \qquad 4y = 12 \quad y = 3$

$x = 6 \qquad z = 3$

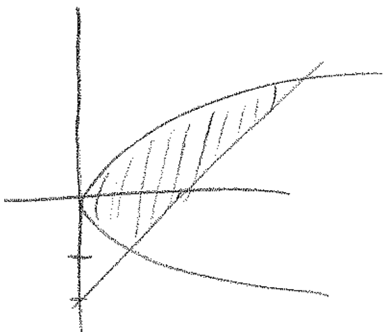
6, 3, 3

-4 if no alg

-3 if alg but not answer

$$\int_1^4 \int_{x-2}^{\sqrt{x}} dy\,dx + \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} dy\,dx$$

(7) 7. Find the area of the region bounded by the curves $x = y^2$, $y = x - 2$ (use double integrals).



$y = y^2 - 2$

$y^2 - x - 2 = 0 \qquad (y-2)(y+1)$

$y = 2,\ y = -1$

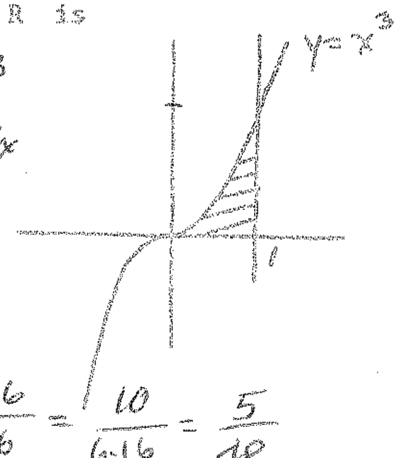
$x = 4,\ y = 1$

$$\int_{-1}^{2}\int_{y^2}^{y+2} dx\,dy = \int_{-1}^{2} y+2-y^2\,dy = \frac{y^2}{2}+2y-\frac{y^3}{3}\Big|_{-1}^{2}$$

a lot missed ?

$$= \frac{4}{2}+4-\frac{8}{3}-\left(\frac12 - 2 + \frac13\right) = +8 - \frac93 - \frac12 = 4\frac12$$

(7) 8. Evaluate the following integral $\iint_R x^2 - xy\ dxdy$ where R is



$$\int_0^1\int_0^{x^3} x^2 - xy\ dy\,dx = \int_0^1 x^2y - \frac{xy^2}{2}\Big|_{y=0}^{x^3} dx$$

$$= \int_0^1 x^5 - \frac{x^7}{2}\,dx = \frac{x^6}{6} - \frac{x^8}{16}\Big|_0^1$$

$$= \frac16 - \frac1{16} - 0 = \cancel{\frac{2}{48} = \frac{1}{24}} = \frac{16-6}{6\cdot 16} = \frac{10}{6\cdot 16} = \frac{5}{48}$$

or $\int_0^1\int_{\sqrt[3]{y}}^{1} x^2 - xy\ dxdy = \int_0^1 \frac{x^3}{3} - \frac{x^2y}{2}\Big|_{y^{1/3}}^{1} = \int_0^1 \frac13 - \frac y2 - \left(\frac y3 - \frac{y^{2/3}}{2}y\right) dy$

(7) 9. Suppose the air pressure in a wind tunnel is given by the function $P(x,y,z) = 20x^2 + y^2 + z^2$ psi. A sensing device at the point $(1,1,1)$ is moved towards the point $(1,0,0)$. At what rate does the pressure change? (Distance is measured in feet. Give units of this rate.) Is it increasing or decreasing?

10 got all pts

$v = (1,0,0) - (1,1,1) = (0,-1,-1)$

$u = (0, -\frac{1}{\sqrt2}, -\frac{1}{\sqrt2})$

$D_uf = \nabla f \cdot u = \langle 40x, 2y, 2z\rangle \langle 0, -1/\sqrt2, -1/\sqrt2\rangle$

$= \langle 40, 2, 2\rangle \langle 0, -1/\sqrt2, -1/\sqrt2\rangle$

$= -4/\sqrt2 = -2\sqrt2$ psi/ft $\boxed{\text{dec}}$

$\int_0^1 \frac13 - \frac56 y + \frac{y^{5/3}}{2}\,dy$

$\frac13 y - \frac56\frac{y^2}{2} + \frac{3y^{8/3}}{16}\Big|_0$

$= \frac13 - \frac5{12} + \frac3{16}$

$16 - 20 + 9 = 5$

2T