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Math 233

Test III

Dec. 12, 1975 Name KEY

DO TEN! 7 and 8 count as one. 15 points each— total of 150.

1. Find the equation of the plane tangent to the surface $z = x^2y + y^2 - 2$ at the point (2,1,3).

$\frac{\partial z}{\partial x} = 2xy \qquad \frac{\partial z}{\partial y} = x^2 + 2y$

at (2,1,3) $\quad 4 \qquad 6$

$\cancel{\nabla z =} 4i + 6j$

$\vec{N} = 4\bar{\imath} + 6\bar{\jmath} - \bar{k}$

$4(x-2) + 6(y-1) - (z-3) = 0$

OR

$w = x^2y + y^2 - 2 - z$

$\frac{\partial w}{\partial x} = 2xy \quad \frac{\partial w}{\partial y} = x^2 + 2y \quad \frac{\partial w}{\partial z} = -1$

$\leftarrow \quad \nabla w = 4\bar{\imath} + 6\bar{\jmath} - \bar{k}$

2. Find an approximate value for $\sqrt{15.5} + 3\sqrt{.95}$.

$\sqrt{16} + 3\sqrt{1} = 7$

$\Delta x = -.5 \quad \Delta y = -.05$

$w = \sqrt{x} + 3\sqrt{y}$

$\frac{\partial z}{\partial x} = \frac{1}{2}x^{-1/2} \qquad \frac{\partial z}{\partial y} = \frac{3}{2}y^{-1/2}$

at (16,1)

$\frac{1}{2\sqrt{16}} = \frac{1}{8} \qquad \frac{3}{2\sqrt{1}} = \frac{3}{2}$

$\Delta w \approx \frac{1}{8}\Delta x + \frac{3}{2}\Delta y = \frac{1}{8}(-.5) + \frac{3}{2}(-.05)$

$= -.0625 - \frac{.15}{2} = -.0625 - .075$

$= -.1375$

$w \approx 7 - .1375 = 6.8625$ or $6\frac{69}{80}$

3. At the point (2,−1,2) what is the direction of maximum decrease for the function $f(x,y,z) = 2xy + yz - z^2$?

$-\nabla f$

$\frac{\partial f}{\partial x} = 2y \qquad \frac{\partial f}{\partial y} = 2x + z \qquad \frac{\partial f}{\partial z} = y - 2z$

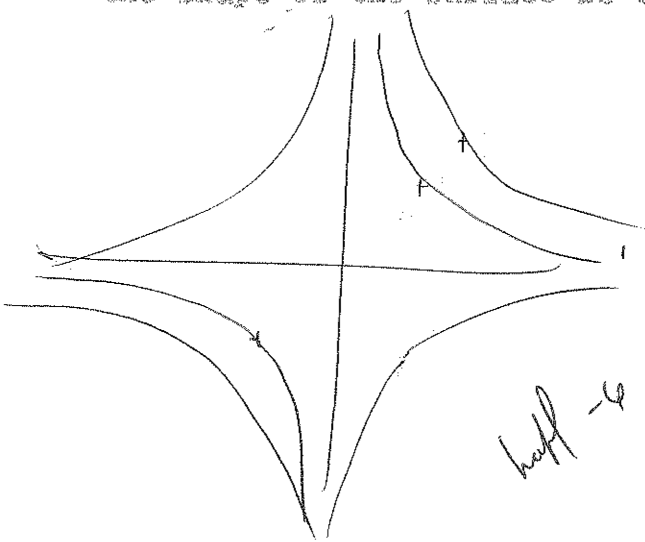
at (2,−1,2)

$-2 \qquad 6 \qquad -5$

$\nabla f = -2\bar{\imath} + 6\bar{\jmath} - 5\bar{k}$

$-\nabla f = 2\bar{\imath} - 6\bar{\jmath} + 5\bar{k}$

4. Sketch several level curves for the surface z = xy (enough so that the shape of the surface is evident).



half -4

5. Justify the following statements (think first, be clear):

a. Every finite decimal is a rational number (can be written as $\frac{p}{q}$, p,q integers).

b. Every repeating decimal is a rational number.

a. $.a_1 a_2 \cdots a_n = \dfrac{a_1 a_2 \cdots a_n}{10^n} \qquad a_i \in \mathbb{Z}$

b. $.\overline{a_1 a_2 \cdots a_n}\cdots = \displaystyle\sum_{n=1}^{\infty} \frac{a_1 \cdots a_n}{10^{kn}} = a_1 \cdots a_n \sum_{n=1}^{\infty} \frac{1}{(10^n)^k}$

$= \dfrac{a_1 \cdots a_n}{10^n} \displaystyle\sum_{n=0}^{\infty} \frac{1}{(10^n)^k}$

$\displaystyle\sum_{n=0}^{\infty} \frac{1}{(10^n)^k} = \dfrac{1}{1-\frac{1}{10^n}}$

$= \dfrac{a_1 \cdots a_n}{\cancel{10^n}} \dfrac{\cancel{10^n}}{10^n - 1}$

6. Find the point(s) on the surface $z = x^2 + y^2$ which is(are) closest to the point $(0,0,9/2)$. What would be the answer if the point is $(0,0,z_0)$?

$$D = d^2 = \sqrt{x^2+y^2+(z-9/2)^2}$$
$$= x^2+y^2+(x^2+y^2-9/2)^2$$
$$\frac{\partial D}{\partial x} = 2x + 2(x^2+y^2-9/2)'(2x)$$
$$= 2x + 4x^3 + 4xy^2 - 18x$$
$$= 4x^3 + 4xy^2 - 16x = 0$$
$$\frac{\partial D}{\partial y} = 2y + 2(x^2+y^2-9/2)'(2y)$$
$$= 2y + 4x^2y + 4y^3 - 18y$$
$$= 4x^2y + 4y^3 - 16y = 0$$

$$4x(x^2+y^2-4) = 0$$
$$4y(x^2+y^2-4) = 0$$

$x = 0 = y$ or $\boxed{x^2+y^2=4}$
circle

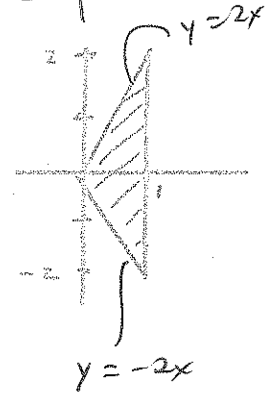
$xy = 0$ $d = \frac{9}{2}$

$x^2+y^2=4$

$4 + (4-\frac{9}{2})^2$

$4\frac{1}{4}$

One Problem

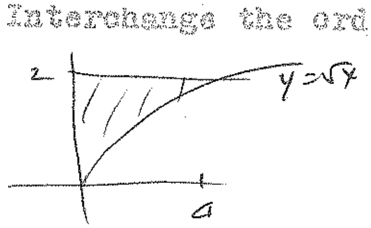
7. Compute $\iint_R f\,dA$ where R is  and $f(x,y) = xy^2$.

$$\int_0^1\int_{-2x}^{2x} xy^2\,dy\,dx$$

$$\int_0^1 \frac{xy^3}{3}\Big|_{-2x}^{2x} dx = \int_0^1 x\frac{(2x)^3}{3} - x\frac{(-2x)^3}{3}\,dx$$

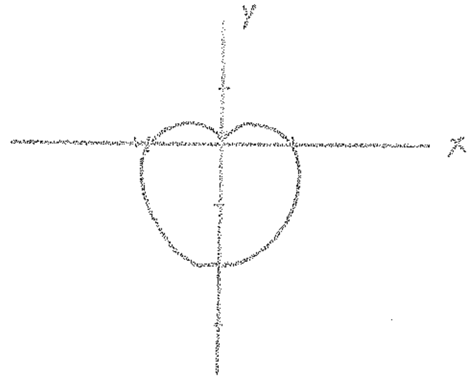
$$= \int_0^1 \frac{8x^4}{3} + \frac{8x^4}{3}\,dx = \int_0^1 \frac{16x^4}{3}\,dx = \frac{16x^5}{15}\Big|_0^1 = \frac{16}{15}$$

8. Interchange the order of integration: $\int_0^4 \int_{\sqrt{x}}^2 f(x,y)\,dy\,dx$



$$\int_0^2\int_0^{y^2} f(x,y)\,dx\,dy$$

9. The region bounded by $r = 1-\sin\theta$ sketched below. By symmetry the x coordinate of the center of mass is $\bar{x} = 0$. Find $\bar{y}$.



$$\iint y \, dA = \int_0^{2\pi}\int_0^{1-\sin\theta} r\sin\theta \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \frac{r^3}{3}\sin\theta \Big|_0^{1-\sin\theta} d\theta$$

$$= \int_0^{2\pi} \frac{(1-\sin\theta)^3 \sin\theta}{3} \, d\theta$$

$$m = \delta \int_0^{2\pi}\int_0^{1-\sin\theta} r \, dr \, d\theta$$

$$\bar{y} = \frac{\displaystyle\int_0^{2\pi}\int_0^{1-\sin\theta} r^2 \sin\theta \, dr \, d\theta}{\displaystyle\int_0^{2\pi}\int_0^{1-\sin\theta} r \, dr \, d\theta}$$

10. Set up only: $\displaystyle\iiint_R f \, dV$ where R is the region in the first octant cut off by the plane through (2,0,0), (0,3,0), and (0,0,1).

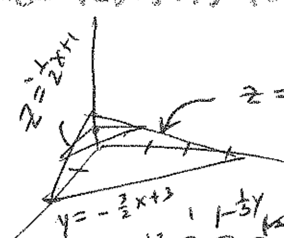
$$Ax + By + Cz + \cancel{D}\,1 = 0$$

$2A + 1 = 0 \qquad A = -\frac{1}{2}$

$3B + 1 = 0 \qquad B = -\frac{1}{3}$

$C + 1 = 0 \qquad C = -1$

$$-\frac{1}{2}x - \frac{1}{3}y - z + 1 = 0$$



$$\int_0^1 \int_0^{1-\frac{1}{3}y} \int^{2-2z-\frac{2}{3}y} f \, dx \, dy \, dz$$

$$3x + 2y + 6z = 6$$

$$2y = 6 - 6z - 3x$$

$$y = 3 - 3z - \frac{3}{2}x$$

$$3x = 6 - 6z - 2y$$

$$x = 2 - 2z - \frac{2}{3}y \qquad z = 1 - \frac{1}{2}x - \frac{1}{3}y$$

11. a. $\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sec^2 x}{1} = 1$

b. $\lim_{n \to \infty} (1+\frac{3}{n})^{2n} = e^6$

$2n \ln(1+3n^{-1})$

$\to\infty \quad \to 0$

$$\lim_{n\to\infty} \frac{\ln(1+3n^{-1})}{1/2n} = \lim_{n\to\infty} \frac{\frac{1}{(1+3/n)} \cdot -\frac{3}{n^2}}{-(2n)^{-2}/2} = \lim_{n\to\infty} \frac{6}{1+3/n} = 6$$

12. Find a polynomial which will approximate cos x for $|x| \le .5$ with an error less than .005 . Write the Taylor series for cos x about $a = \frac{\pi}{2}$.

		a=0	a=π/2
y	$\cos x$	1	0
y'	$-\sin x$	0	−1
y''	$-\cos x$	−1	0
y'''	$\sin x$	0	1
y^{iv}	$\cos x$	1	0
	⋮		−1

$1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} \pm \cdots$

$|R_n| \le \frac{|x|^{n+1}}{(n+1)!} \le \frac{(.5)^{n+1}}{(n+1)!}$

$n = \emptyset \quad en \le \frac{.5}{2!} = .25$

$n = 3 \quad en \le \frac{(.5)^4}{4!} = \frac{.0625}{24}$

$= .0026$

$\cos x \approx 1 - \frac{x^2}{2}$

$24\overline{)625}$ = 26; 48; 145

$$\sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2k+1}}{(2k)!}$$

13. Test for convergence, justify.

a. $\sum_{k=1}^{\infty} \frac{2^k}{3^{2k}}$ b. $\sum_{k=1}^{\infty} \frac{k-1}{k^4+5}$ c. $\sum_{k=1}^{\infty} \frac{k!}{k^2}$

$\sum_{k=1}^{\infty} \left(\frac{2}{9}\right)^k$

$r = \frac{2}{9} < 1$

geometric converges

$\frac{k-1}{k^4+5} < \frac{k}{k^4} = \frac{1}{k^3}$

$\sum \frac{1}{k^3}$ conv $p=3$

CT $\Rightarrow \sum \frac{k-1}{k^4+5}$ conv

$\frac{(k+1)!}{(k+1)^2} \Big/ \frac{k!}{k^2}$

$= \frac{k+1}{(k+1)^2} k^2$

$= \frac{k^2}{k+1} \to \infty > 1$

RT $\Rightarrow$ div

14. What do each of these series converge to?

a. $\sum_{k=1}^{\infty} \frac{3^k}{4^{2k}}$ b. $\sum_{k=1}^{\infty} \frac{3}{k!} = 3 \sum_{k=0}^{\infty} \frac{3}{k!} - 3 = 3e^{3} - 3$

$\sum_{k=1}^{\infty} \left(\frac{3}{16}\right)^k$

$= \sum_{k=0}^{\infty} \left(\frac{3}{16}\right)^k - 1$

$= \frac{1}{1 - 3/16} - 1$

$= \frac{1}{13/16} - 1$

$= \frac{16}{13} - 1$

$= \frac{3}{13}$