Math 233
Final Exam
June 9, 1978

Name KEY

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25 min
with [illegible]

Show Work!

(10) 1. Find:

a. $\lim_{n\to\infty} \frac{n-n^3}{1+n^2-2n^3} =$ $\frac{\frac{1}{n^2}-1}{1/n^3+\frac{1}{n}-2} \to \frac{1}{2}$

b. $\lim_{x\to 0} \frac{\cos x - 1}{x} =$ $\frac{-\sin x}{1} \to 0$

(10) 2. Find the equation of the plane tangent to the surface $x^3y + xyz = 0$ at $(1,1,-1)$.

$w =$ ↑ $\nabla w = (3x^2+yz)\bar{i} + (x^3+xz)\bar{j} + xy\bar{k}$

$(1,1,-1)$ $\vec{N} = 2\bar{i} + 0\bar{j} + k = 2\bar{i}+\bar{k}$

$2(x-1) + (z+1) = 0$

$2x + z - 1 = 0$

(12) 3. For $f(x,y,z) = x\cos z - y\sin x$, find:

a. ∇f

b. directional derivative in the direction of $2\bar{i} - 3\bar{j} + \bar{k}$ at $(\pi,1,0)$.

$\|\cdot\| = \sqrt{4+9+1} = \sqrt{14}$

a. $\nabla f = (\cos z - y\cos x)\bar{i} - \sin x\,\hat{j} - x\sin z\,\bar{k}$

b. $\vec{u} = \frac{2}{\sqrt{14}}\bar{i} - \frac{3}{\sqrt{14}}\bar{j} + \frac{1}{\sqrt{14}}k$

at $(\pi,1,0)$ $\nabla f = \cos 0 - \cos\pi\,\bar{i} - \sin\pi\,\bar{j} - \pi\sin 0\,\bar{k} = 2\bar{i}$

$\frac{df}{d\bar{u}} = \nabla f \cdot \bar{u} = 2\bar{i}\cdot\bar{u}$

$= \frac{4}{\sqrt{14}}$

(8) 4. What is the direction of maximum increase for the function $f(x,y,z) = x^2 + y$ at $(1,2,1)$?

$\nabla f = 2x\bar{i} + \bar{j} + 0k$

at $(1,2,1)$ $2\bar{i} + \bar{j}$

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(15) 5. Approximate $(1.01)e^{.02}$. $z = xe^y$

$x=1 \quad y=0$

$\Delta x = .01 \quad \Delta y = .02$

$\frac{\partial z}{\partial x} = e^y \quad \frac{\partial z}{\partial y} = xe^y$

$\Delta z_{tan} = e^0(.01) + 1e^0(.02)$

$= .01 + .02 = .03$

$z \approx 1e^0 + .03 = 1.03$

(15) 6. Find the area of the region bounded by $y = -x$ and $x = y - y^2$.

$-y = y - y^2$

$y^2 - 2y = 0$

$y(y-2) = 0$

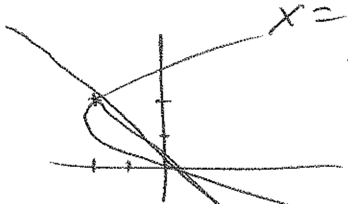
$y = 0, 2$

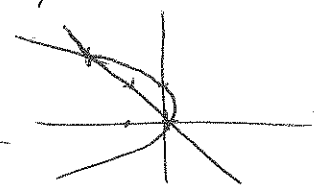
$x = 0, -2$

$\int_0^2 \int_{-y}^{y-y^2} dx\, dy = \int_0^2 2y - y^2\, dy$

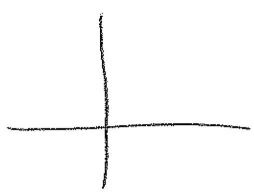
$= y^2 - \frac{y^3}{3}\Big|_0^2 = 4 - \frac{8}{3}$

$= \frac{4}{3}$





(10) 7. Set up the integral for the area bounded by the rose curve $r = \sin 2\theta$.



$\int_0^{2\pi} \int_0^{\sin 2\theta} r\, dr\, d\theta$

$\int_0^{2\pi}$ (7)

(15) 8. a. Compute $\int_0^3 \int_0^{3y} xy\, dx\, dy$.

F/9

9

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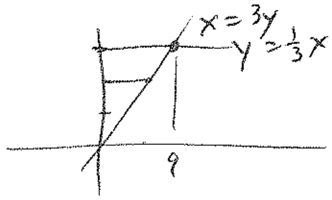
b. Write as an integral with order of integration reversed.

a. $\int_0^3 \frac{x^2 y}{2}\Big|_0^{3y} dy = \int_0^3 \frac{(3y)^2 y}{2}\, dy = \int_0^3 \frac{9y^3}{2}\, dy = \frac{9y^4}{8}\Big|_0^3 = \frac{729}{8}$

b.



$\int_0^9 \int_{\frac{1}{3}x}^{3} xy\, dy\, dx$

(4)

(1)

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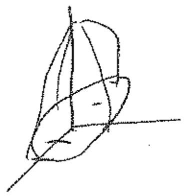
(10) 9. <u>Set up</u> the integral for the volume bounded by $z = 4 - x^2 - 4y^2$ and the xy plane.



$z = 0 \quad x^2 + 4y^2 = 4$

$$\int_0^1 \int_{-\sqrt{4-4y^2}}^{\sqrt{4-4y^2}} \int_0^{4-x^2-4y^2} dz\, dx\, dy$$

(15) 10. Express as a fraction of integers: $2.3461\overline{461}\ldots$

$$2.3\cancel{4} + 461 \sum \left[10^{-5} + 10^{-8} + \cdots \right]$$

$$2.3 + \frac{461}{10^{5}} \sum_{k=0}^{\infty} \frac{1}{10^{3k}} = 2.3 + \frac{461}{10^5} \frac{1}{1 - 1/1000} \qquad \frac{999}{1000}$$

$$= 2.3 + \frac{461}{10^2 \, 999}$$

$$\frac{23}{10} \qquad = \frac{229770 + 461}{99900}$$

$$= \cancel{\frac{230231}{99900}}$$

$$\boxed{\frac{23438}{9990}}$$

23
999
207
207
207
229770
461
230231

22977
461
23438

(35) 11. Test for convergence, give reasons.

a. $\sum_{k=1}^{\infty} \frac{3}{k^{3/2}}$ $\qquad p = 3/2 > 1$ conv.

b. $\sum_{k=1}^{\infty} \frac{k}{k+3}$ $\qquad \frac{k}{k+3} \to 1$ div

c. $\sum_{k=1}^{\infty} \frac{k}{k^3+k+3}$ $\qquad \frac{k}{k^3+k+3} \le \frac{k}{k^3} = \frac{1}{k^2}$ CT conv.

d. $\sum_{k=1}^{\infty} \frac{k}{5^k}$ $\qquad$ RT $\qquad \frac{\frac{k+1}{5^{k+1}}}{k/5^k} = \frac{k+1}{k} \cdot \frac{1}{5} \to \frac{1}{5} < 1$ conv.

e. $\sum_{k=1}^{\infty} \frac{2^k}{5^k+1}$ $\qquad \frac{2^k}{5^k+1} \le \frac{2^k}{5^k} = \left(\frac{2}{5}\right)^k$ CT $\frac{2}{5} < 1$ conv.

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(20) 12. Do the following series converge, and if so to what?

a. $\sum_{k=1}^{\infty} \frac{2}{4^{k+1}}$ b. $\sum_{k=1}^{\infty} (1.1)^k$ c. $\sum_{k=1}^{\infty} \frac{(-2)^k}{3^{k+2}}$

$\frac{2}{4^2} \sum_{n=0}^{\infty} \frac{1}{4^n}$

$\frac{2}{4^2} \frac{1}{1-1/4}$

$\frac{2}{4^2} \cdot \frac{4}{3} = \frac{2}{6}$

$1.1 > 1$

div.

$\frac{1}{9} \sum_{k=1}^{\infty} \left(-\frac{2}{3}\right)^k$

$-\frac{2}{3} \cdot \frac{1}{9} \sum_{k=0}^{\infty} \left(-\frac{2}{3}\right)^k$

$-\frac{2}{27} \cdot \frac{1}{1+2/3}$ $5/3$

$-\frac{2}{27} \cdot \frac{3}{5} = -\frac{2}{45}$

(15)

13. Write out (and simplify) the first 4 terms of the Taylor Series for $f(x) = \tan x$ about $a = \frac{\pi}{4}$.

$f'(x) = \sec^2 x$

$f''(x) = 2\sec x(\sec x \tan x)$

$= 2\sec^2 x \tan x$

$f'''(x) = 2\sec^2 x \sec^2 x$

$+ 4\sec x \sec x \tan x \tan x$

$= 2\sec^4 x + 4\sec^2 \tan^2 x$

$f(\frac{\pi}{4}) = 1$

$f'(\frac{\pi}{4}) = 2$

$f''(\frac{\pi}{4}) = 2 \cdot 2 \cdot 1 = 4$

$f'''(\frac{\pi}{4}) = 2 \cdot 2 \cdot 2^2 + 4 \cdot 2 \cdot 1$

$= 16$

$\tan x = 1 + 2(x-\frac{\pi}{4}) + \frac{4}{2}(x-\frac{\pi}{4})^2$ $+ \frac{16}{6}(x-\frac{\pi}{4})^3$

$= 1 + 2(x-\frac{\pi}{4}) + 2(x-\frac{\pi}{4})^2 + \frac{8}{3}(x-\frac{\pi}{4})^3$

(5) 14. How many terms(give the highest power of x) are needed in the MacLaurin series for $f(x) = \cos x$ so that the error is less than .00001 for $|x| \leq .1$.

$|R_n| \leq \frac{|x|^{n+1}}{(n+1)!} \leq \frac{(.1)^{n+1}}{(n+1)!}$

$n=0$ $\frac{.1}{1} = .1$

$n=1$ $\frac{.01}{2!} = .05$

$n=2$ $\frac{.001}{6} =$

$n=3$ $\frac{.0001}{24} = .000004$

$n=4$ $\frac{.00001}{120}$

3

4

45