

MATH 332  
FINAL EXAM  
DECEMBER 9, 1981

NAME KEY  
S-U? no

191 +4

TIME 22:30.

OK

First left  
after 1:10, +1:30

Show complete solutions with justifications.

ON/17 SAME problem up to 15pts & could  
have been left on

(21+) 1. Let  $\bar{u} = (1, 2, -1, 0)$ ,  $\bar{v} = (2, 2, 3, -1)$ . Find:

a.  $\bar{u} \cdot \bar{v} = 1 \cdot 2 + 2 \cdot 2 + (-1)(3) + 0(-1)$   
 $= 2 + 4 - 3 + 0 = \boxed{3}$

b.  $2\bar{u} + 3\bar{v} = (2, 4, -2, 0) + (6, 6, 9, -3)$   
 $= \boxed{(8, 10, 7, -3)}$

c.  $\|\bar{u}\| = \sqrt{1^2 + 2^2 + (-1)^2 + 0^2} = \boxed{\sqrt{6}}$

d. a unit vector in the direction of  $\bar{v}$ .

$\|\bar{v}\| = \sqrt{4+4+9+1} = \sqrt{18} = 3\sqrt{2}$

$\boxed{\left( \frac{2}{3\sqrt{2}}, \frac{2}{3\sqrt{2}}, \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}, -\frac{1}{3\sqrt{2}} \right)} \text{ or } \left( \frac{3}{\sqrt{18}}, \dots \right)$

e. Are  $\bar{u}$  and  $\bar{v}$  linearly independent?

~~all 1, 2, -1~~ yes, not multiple

YES

f. Is  $(1, 1, 1, 1)$  in the span of  $\bar{u}$  and  $\bar{v}$ ?

$(1, 1, 1, 1) = a(1, 2, -1, 0) + b(2, 2, 3, -1)$

$a + 2b = 1 \quad -a + 3b = 1$

$2a + 2b = 1$

$-b = 1$

$b = -1$

$-a = 1 - 3(-1)$

$= 4$

$a = -4$

$-4 + (-2) \notin 1$

g. What is the dimension of the span of  $\bar{u}$  and  $\bar{v}$ ? 10

2.

(9) 2. Find each determinant. Look for short ways. Justify!

a.  $\begin{vmatrix} 2 & 2 & 3 & 1 \\ -1 & 3 & 0 & 1 \\ 2 & 3 & 8 & 0 \\ 4 & 4 & 6 & 2 \end{vmatrix} = 0$

$$\det A = \begin{vmatrix} 2 & 1 & 3 & 7 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -20$$

(6) 3. If  $\det A = 5$ , find:  $A_{4 \times 4}$

a.  $\det A^T = 5$

b.  $\det A^{-1} = \frac{1}{5}$

c.  $\det 2A = 2^4(5) = 80$

(15) 4. Perform the following computations:

a.  $\begin{bmatrix} 1 & 2 & 5 \\ 4 & 5 & 3 \\ 2 & 7 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 7 \\ 5 & 3 & 6 \end{bmatrix}$

b.  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \cdot \frac{1}{7-15} \begin{bmatrix} 7 & -3 \\ -5 & 1 \end{bmatrix} = \frac{1}{-8} \begin{bmatrix} 7-3 \\ -5+1 \end{bmatrix} = \begin{bmatrix} 7/8 & 3/8 \\ 5/8 & -1/8 \end{bmatrix}$

c.  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 & 2 \\ 3 & 2 & 1 \\ 5 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1+4+2 \\ 3+2 \\ 5+1 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 3 \end{bmatrix}$

d.  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{not pass}$

e.  $\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = - \begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix} = - [5-9] = 4$

(b) 5. Give the elementary matrix which performs each of the following elementary row operations on a  $4 \times 4$  matrix.

a. adds twice row 3. to row 4.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

b. Interchanges rows 2 and 3

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(5) 6. Is the matrix  $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 0 & 1 \\ 6 & 3 & 8 \end{bmatrix}$  invertible?

$$\det = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 0 & 1 \\ 6 & 3 & 8 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 5 & 0 \\ 6 & 3 \end{vmatrix} = 0 + 12 - 45 - 0 - 3 - 30 < 0$$

yes

(5) 7. Does the system  $2x_1 + x_2 + x_3 = 0$

$$3x_1 + 2x_2 + x_3 - 3x_4 = 0$$

$$x_2 = 0$$

$$2x_1 + x_2 + 7x_4 = 0$$

have solutions other than  $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0$ ?

$$\left| \begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 3 & 2 & 1 & -3 \\ 0 & 1 & 2 & 0 \\ 2 & -1 & 0 & 7 \end{array} \right| = - \left| \begin{array}{cc|cc} 2 & 1 & 0 & 2 \\ 3 & 1 & -3 & 1 \\ 0 & 0 & 7 & 0 \end{array} \right| = - \begin{bmatrix} 2 & 1 & 0 & 2 \\ 3 & 1 & -3 & 1 \\ 0 & 0 & 7 & 0 \end{bmatrix} = - \begin{bmatrix} 14 & -6 & 0 \\ -9 & -21 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \neq 0$$

No.

(d) 8. Suppose the  $3 \times 4$  matrix A has rank 2. What is the dimension of the solution space of  $Ax = 0$ ?

$$\begin{bmatrix} & \end{bmatrix} \begin{array}{l} \mathbb{R}^4 \rightarrow \mathbb{R}^3 \\ n=4 \end{array}$$

$$2 + N = 4$$

$$N = 2$$

(10) Q. Find the inverse of the matrix

$$\left[ \begin{array}{ccc|cc} 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 3 & 0 & 1 \\ 1 & 2 & 5 & 0 & 0 \end{array} \right]$$

$$\left[ \begin{array}{ccc|cc} 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 3 & 0 & 1 \\ 1 & 2 & 5 & 0 & 0 \end{array} \right]$$

$$\left[ \begin{array}{ccc|cc} 6 & 0 & 1 & -6 & 0 \\ 0 & 1 & 1 & -2 & 1 \\ 0 & 0 & 1 & 3/2 & -1/2 \end{array} \right]$$

$$\left[ \begin{array}{ccc} -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 2 & -\frac{1}{2} \\ \frac{3}{2} & -1 & \frac{1}{2} \end{array} \right]$$

$$\text{①-2} \left[ \begin{array}{ccc|cc} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 1 \\ 0 & 2 & 4 & -1 & 0 \end{array} \right] \text{②-3} \left[ \begin{array}{ccc|cc} 1 & 0 & 0 & -4/2 & 1/2 \\ 0 & 1 & 0 & 7/2 & 2/2 \\ 0 & 0 & 1 & 3/2 & -1/2 \end{array} \right]$$

$$\text{③-2} \left[ \begin{array}{ccc|cc} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 1 \\ 0 & 0 & 2 & 3 & -2 \end{array} \right]$$

$$\left[ \begin{array}{ccc} \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{7}{2} & -2 & \frac{1}{2} \\ \frac{3}{2} & -1 & \frac{1}{2} \end{array} \right] \times$$

$\frac{3}{2} + 2$

(6) 10. Do the following sets of vectors form a basis for  $\mathbb{R}^3$ ?

a.  $(1, 2, 1), (1, -1, 1), (1, -2, -3)$

$$\left| \begin{array}{cc|c} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & -2 & -3 \end{array} \right| = 1(1 - 2) - (-1 - 2 - 6) = 3 + 2 - 2 = 3 \neq 0 \text{ Yes}$$

b.  $(1, 1, 1), (2, 1, 0)$

No, only 2

$$\overline{\left| \begin{array}{ccc} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & -2 & -3 \end{array} \right|}$$

$$\left[ \begin{array}{ccc} 1 & 2 & 1 \\ 0 & -3 & 0 \\ 0 & -4 & -4 \end{array} \right]$$

(3) 11. Show:

If  $T$  is a linear transformation of  $V$  into  $W$ , then  $T(-v) = -T(v)$  for any  $v$  in  $V$ .

$$-v = (-1)v$$

$$T(-v) = T((-1)v)$$

$$= (-1)T(v)$$

$$\left[ \begin{array}{ccc} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & 0 & -3 \end{array} \right]$$

$$\left[ \begin{array}{ccc} 1 & 1 & 1 \\ 0 & -3 & -4 \\ 0 & 0 & -4 \end{array} \right]$$

10/45

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lot of trouble on  
this page

- (10+) 12. Determine if each of the following subsets of  $\mathbb{P}_2$  is a subspace. If so, find its dimension. Justify!

a. all  $p(x) = a_0 + a_1x + a_2x^2$  where  $a_0 \neq 0$ .

$0$  poly not in.

also  $\frac{1+x}{-1+x} = 2x$  not in

b. all  $p(x) = a_0 + a_1x + a_2x^2$  where  $a_0 = a_1 = a_2$ .

$$(a+ax+ax^2) + (b+bx+bx^2) \\ = a+b + (a+b)x + (a+b)x^2 \text{ in}$$

$$k(a+ax+ax^2) = (ka+ka)x + (ka)x^2 \text{ in}$$

Ys

- (7) 13. Find a basis for the subspace of  $\mathbb{R}^4$  spanned by the vectors  $(1, 2, 3, 7)$ ,  $(1, 0, 1, 2)$  and  $(1, 3, 4, 12)$ . What is the dimension?

$$\begin{bmatrix} 1 & 2 & 3 & 7 \\ 1 & 0 & 1 & 2 \\ 1 & 3 & 4 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 7 \\ 0 & -2 & -2 & -5 \\ 0 & 1 & 1 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 7 \\ 0 & 1 & 1 & 5 \\ 0 & -2 & -2 & -5 \end{bmatrix}$$

$$\textcircled{3} + 2\textcircled{2} \begin{bmatrix} 1 & 2 & 3 & 7 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad \dim 3 \quad \text{only } \textcircled{1}$$

- (5) 14. Show that the zero vector of a vector space is unique. Use only the axioms.

$$\bar{z}, \bar{0}$$

$$\bar{z} + \bar{0} = \bar{0}$$

$$0 + \bar{z} = \bar{z}$$

$$\bar{0} = \bar{z}$$

done well

- (10) 15. Which of the following functions from  $M_{2,2}$  into  $\mathbb{R}$  are linear transformations? Justify!

a.  $F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a - c + b$

$$F\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + F\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = a_1 - c_1 + b_1 + a_2 - c_2 + b_2$$

$$F\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}\right) = (a_1+a_2) - (c_1+c_2) + b_1 + b_2 =$$

$$F(k\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = F\left(\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}\right) = ka - kc + kb = k(a - c + b)$$

b.  $F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$

yes

$$\begin{aligned} F(k\begin{bmatrix} a & b \\ c & d \end{bmatrix}) &= k a k d - k b k c \\ &= k^2(ad - bc) = k^2 F\begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{aligned}$$

not

- (10) 16. Carefully state the following definitions:

- a. A function  $f: V \rightarrow W$  is a linear transformation

$$\text{if } f(u+v) = f(u) + f(v) \text{ all } u, v$$

$$f(ku) = kf(u) \text{ all } k, u$$

- b. Null space, or kernel of a linear transformation  $T$

$$\text{Set of } u \in V \text{ s.t. } T(u) = 0$$

- (5) 17. Let  $T: \mathbb{R}_2 \rightarrow \mathbb{R}_2$  be a linear transformation. If  $T(1) = x$ ,  $T(x) = x + 1$ ,

$$T(x^2) = x^2 - 2, \text{ find } T(x^2 + 3x + 4).$$

$$= T(x^2) - 3T(x) + 4T(1)$$

$$= (x^2 - 2) - 3(x + 1) + 4(1)$$

$$= x^2 - 2 - 3x - 3 + 4x$$

$$= x^2 + x + 2$$

25

16.145

(3) 18. Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be a linear transformation. If  $T(1,0) = (1,1,2)$  and  $T(0,1) = (2,0,3)$ , find the matrix A such that  $T(x) = Ax$ .

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}$$

(16) 19. Find the standard matrix for each of the following linear transformations of  $\mathbb{R}^2$  into  $\mathbb{R}^2$ .

a.  $T(x_1, x_2) = (x_1 + x_2, 2x_1 - x_2)$

$$T(0,0) = (0,0) \quad T(0,1) = (1,-1)$$

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

b. rotation of  $45^\circ$  clockwise about the origin.



$$\begin{bmatrix} y_{12} & y_{22} \\ -y_{12} & y_{22} \end{bmatrix}$$

c. orthogonal projection onto the y-axis.

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

d. reflection about the line  $y = -x$ .



$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

e.  $T(x_1, x_2) = 4(x_1, x_2)$ .

$$\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

18:30

(3) 20. Let linear transformation T have the following standard matrix:

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 2 & 6 \\ 1 & 3 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 2 & 6 \\ 1 & 3 & 5 \end{bmatrix}$$

a. Find a basis for the null space (kernel).

$$\begin{pmatrix} 2 & 1 & 1 \end{pmatrix}$$

b. Find a basis for the range.

$$\textcircled{2} - 2\textcircled{1} \begin{bmatrix} 1 & 2 & 4 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

c. What are the rank and nullity?

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 4 & 6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_3 = t$$

$$x_2 = -t$$

$$x_1 = -2(-t) - 4(t) \\ = 2t - 4t = -2t$$

$$t \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

$$\textcircled{b} \quad \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$$\textcircled{c} \quad \begin{array}{l} \text{rank} = 2 \\ \text{null} = 1 \end{array}$$

35  
X

20:30

(15). 21. Let  $T$  be a linear transformation with standard matrix  $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 3 & 1 \end{bmatrix}$

a.  $T: \mathbb{R}^4 \rightarrow \mathbb{R}^2$  (range)

b. Is  $(1,1)$  in the range of  $T$ ? ~~not in  $\mathbb{R}^4$~~

c. Is  $(1,1)$  in the kernel (null space) of  $T$ ? ~~not in  $\mathbb{R}^4$~~

$\textcircled{b}$   $\textcircled{c} \quad \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} ?$

$$\left[ \begin{array}{cc|c} 1 & 0 & 2 & 1 \\ 2 & 1 & 3 & 1 \end{array} \right] \xrightarrow{\text{R}_2 - 2\text{R}_1} \left[ \begin{array}{cc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & -1 \end{array} \right]$$

Not unique sol.

OR  
Basis for range:

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 3 \\ 0 & 1 \end{bmatrix}$$

$$\left[ \begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{array} \right] \xrightarrow{\text{R}_2 + \text{R}_3, \text{R}_3 + \text{R}_4} \left[ \begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$k_1(1,2) + k_2(0,1) \stackrel{?}{=} (1,1)$$

$$k_1 = 1 \\ 2k_1 + k_2 = 1$$

$$k_2 = -1 \\ 22:30$$

d. What are the rank and nullity?

$$\text{rank} = 2 \quad \text{null} = 2$$

OR  
 $\text{R}_2 - 2\text{R}_1$

$$\left[ \begin{array}{cc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & -1 \end{array} \right]$$

$$s(-2,1,1,0) + t(-1,1,0,1) = (1,1,1) ?$$

$$\begin{aligned} x_4 &= t \\ x_3 &= s \\ x_2 &= s+t \\ x_1 &= -2s-t \\ &= -ts-t \end{aligned}$$

$$\left[ \begin{array}{cc|c} -2 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{array} \right]$$

inconsistent  
no!