

MATH 332  
Final Exam  
December 14, 1989

X 128.4 (75.5%)

70

Name Key

Time OK  
Started to leave at  
1:17. Mid 1:48  
all but 3 gone at 2:00

Exam double weight? \_\_\_\_\_

1. Complete the following DEFINITIONS: (20)

a. The vectors  $v_1, v_2, \dots, v_n$  are linearly independent and only if

b. The vectors  $v_1, v_2, \dots, v_n$  form a basis for the vector space V iff

c. The dimension of a vector space is

all

d. The null space(kernel) of a linear transformation  $T: V \rightarrow W$  is

2. Complete and prove: (40)

a. If the matrix Q is orthogonal, then  $\det Q = \underline{\pm 1}$ .

$$Q^T Q = I$$

$$\det Q^T Q = \det I = 1$$

$$\det Q^T \det Q = 1$$

$$(\det Q)^2 = 1 \quad \det Q = \pm 1$$

b. If A is invertible, then  $Ax = 0$  has only the solution  $x = 0$ .

$$Ax = 0$$

$$A^{-1}Ax = A^{-1}0$$

$$Ix = 0$$

$$x = 0$$

- c. If the vectors  $v_1, v_2, \dots, v_n$  are linearly dependent, then at least one can be written as a linear combination of the others.

In dep  $\Rightarrow \exists c_1, \dots, c_n$  not all zero s.t.

$$c_1 v_1 + \dots + c_n v_n = 0$$

anso  $c_i \neq 0$ . Then

$$c_1 v_1 = -c_2 v_2 - \dots - c_n v_n$$

$$v_1 = -\frac{c_2}{c_1} v_2 - \dots - \frac{c_n}{c_1} v_n$$

~~ODD~~

- d. The range of the linear transformation  $T: V \rightarrow W$  is a subspace of W.

$$R = \{y \mid T(x) = y \text{ for some } x \in V\}$$

Let  $y_1, y_2$  be in  $R$ . so  $y_1 = T(x_1), y_2 = T(x_2)$

$$y_1 + y_2 = T(x_1) + T(x_2) = T(x_1 + x_2), x_1 + x_2 \in V.$$

so  $y_1 + y_2 \in R$

If  $y$  in  $R$ , then  $y = T(x)$  for some  $x \in V$

$$ky = kT(x) = T(kx), kx \in V$$

so  $R$  is a subspace of  $W$ .

3. Are the following vectors in  $\mathbb{R}^4$  linearly independent? Justify. (10)

$$(1, 2, 1, 2), (1, 1, 0, 0), (2, -1, 0, 1), (0, 1, 0, 1)$$

all

$$c_1(1, 2, 1, 2) + c_2(1, 1, 0, 0) + c_3(2, -1, 0, 1) + c_4(0, 1, 0, 1) = (9, 0, 0, 0)$$

$$\det \begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = -2 - 2 = -4 \neq 0 \text{ Yes.}$$

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & -1 & -5 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & -2 & -3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & -1 & -5 & 1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 7 & -1 \end{bmatrix}$$

4. For the following matrix:

(20)

$$A = \begin{bmatrix} 1 & 2 & 2 & 0 & -9 & 7 & 0 & 1 \\ 0 & 0 & 2 & 7 & 2 & 6 & 0 & 1 \\ 0 & 0 & 0 & 3 & 6 & 7 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 \end{bmatrix}$$

- a. The row space is a 4 dimensional subspace of  $\mathbb{R}^8$ .
- b. The column space is a 4 dimensional subspace of  $\mathbb{R}^4$ .
- c. The rank is 4.
- d. The null space of  $T(x) = Ax$  is a 4 dimensional subspace of  $\mathbb{R}^8$ .

(e) Find a basis for the solution space of  $Ax = 0$ .

too hard

$$\begin{aligned} X_8 &= t & X_7 &= s, & X_6 &= w & X_5 &= \\ X_5 + w + 2s + 3t &= 0 & X_5 &= -w - 2s - 3t \\ X_5 + 6v + 7w + s &= 0 & X_5 &= -6v - 7w - s \\ 3X_4 + 6\left(\frac{-w - 2s - 3t}{-6v - 7w - s}\right) + 7w + s &= 0 \\ 3X_4 &= -w + 11s + 18t \end{aligned}$$

$$(9, 0, -37, 6, 3, 0, 0, 1) (6, 0, -12, 4, -3, 0, 0, 0) \\ (2, 0, -9, 2, -1, 1, 0, 0) (-2, 1, 0, 0, 0)$$

5. Consider the set of all matrices of the form (for any real numbers a,b,c)

$$\begin{bmatrix} a & b \\ -b & c \end{bmatrix}$$

(15?)

3 a. Give an example of such a matrix

$$\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$$

only if

6 b. Verify that this set is a subspace of  $M_{22}$ .

$$\begin{bmatrix} a & b \\ -b & c \end{bmatrix} + \begin{bmatrix} d & e \\ -e & f \end{bmatrix} = \begin{bmatrix} a+d & b+e \\ -b+e & c+f \end{bmatrix} \quad \text{some form}$$

$$k \begin{bmatrix} a & b \\ -b & c \end{bmatrix} = \begin{bmatrix} ka & kb \\ -kb & kc \end{bmatrix} \quad \text{so closed}$$

6 c. Find a basis for this subspace. What is the dimension?

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad 3$$

6. Find the least squares line for the data:

$$\begin{array}{c|c} x & y \\ \hline 1 & 1 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{array} \quad M = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \quad M^T M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \quad M^T B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 15 \\ 39 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 15 \\ 39 \end{bmatrix} \quad \begin{bmatrix} b \\ m \end{bmatrix} = \frac{1}{100+20} \begin{bmatrix} 30 - 10 \\ -10 + 15 \end{bmatrix} \begin{bmatrix} 15 \\ 39 \end{bmatrix}$$

$$= \frac{1}{120} \begin{bmatrix} 0 \\ 20 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.3 \end{bmatrix} \quad (10) \quad \frac{156}{-120} \quad \frac{126}{20}$$

7. Find

$$\det \begin{bmatrix} 2 & 4 & 2 & 0 & 0 \\ -2 & -3 & -1 & 0 & 0 \\ 4 & 8 & 5 & 1 & 2 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} = 2 \det \begin{bmatrix} 2 & 4 & 2 & 0 \\ -2 & -3 & -1 & 0 \\ 4 & 8 & 5 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad (63)$$

$$= 2 \left[ -\det \begin{bmatrix} 2 & 4 & 2 \\ -2 & -3 & -1 \\ 0 & 1 & 1 \end{bmatrix} + \det \begin{bmatrix} 2 & 4 & 2 \\ -2 & -3 & -1 \\ 4 & 8 & 5 \end{bmatrix} \right] = 2[-(-4)] \quad (4)$$

8. If the linear transformation  $T((x,y))$  is a rotation clockwise of  $\pi/4$  about  $(0,0)$ , write the matrix form for  $T$ . (5)

$$\begin{bmatrix} x_2 & y_2 \\ -x_2 & y_2 \end{bmatrix}$$

$\frac{1}{2}$

9. Find the inverse of the matrix (if any):

$$\left[ \begin{array}{ccc|cc} 2 & 3 & 1 & 1 & 0 & 0 \\ 4 & 7 & 3 & 0 & 1 & 0 \\ -6 & -7 & 0 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|cc} 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 2 & 3 & 3 & 0 & 1 \end{array} \right]$$

(10)

all

$$\left[ \begin{array}{ccc|cc} 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|cc} 2 & 3 & 0 & -6 & 2 & -1 \\ 0 & 1 & 0 & -9 & 3 & -1 \\ 0 & 0 & 1 & 7 & -2 & 1 \end{array} \right]$$

$$\left[ \begin{array}{ccc|cc} 2 & 0 & 0 & 21 & -2 & 2 \\ 0 & 1 & 0 & -9 & 3 & -1 \\ 0 & 0 & 1 & 7 & -2 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|cc} 1 & 0 & 0 & \frac{21-2}{2} & 1 & 1 \\ 0 & 1 & 0 & -9 & 3 & -1 \\ 0 & 0 & 1 & 7 & -2 & 1 \end{array} \right]$$

$$\begin{bmatrix} \frac{19}{2} & -2 & 1 \\ -9 & 3 & -1 \\ 7 & -2 & 1 \end{bmatrix}$$

10. Find a basis for the space spanned by the vectors  $(2,1,1,0,1)$ ,  $(4,3,5,0,2)$ ,  $(-2,2,2,1,-3)$ ,  $(0,1,1,0,0)$ . What is the dimension of this subspace?(10)

$$\begin{bmatrix} 2 & 1 & 1 & 0 & 1 \\ 4 & 3 & 5 & 0 & 2 \\ -2 & 2 & 2 & 1 & -3 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 & 0 & 1 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 3 & 3 & 1 & -3 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 & 0 & 1 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & -6 & 1 & -2 \\ 0 & 0 & -2 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 & 0 & 1 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & -2 & 1 & -2 \\ 0 & 0 & -6 & 0 & 0 \end{bmatrix} \xrightarrow{\text{few}} \begin{bmatrix} 2 & 1 & 1 & 0 & 1 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & -2 & 1 & -2 \\ 0 & 0 & 0 & -2 & 4 \end{bmatrix}$$

few

11. Show that the following is a linear transformation from  $M_{22}$  to  $P_3$ :(10)

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ax^3 + bx^2 + cx + d$$

$\frac{1}{2}$

$$T\left(\begin{bmatrix} ay \\ cd \end{bmatrix} + \begin{bmatrix} ex \\ fg \end{bmatrix}\right) = T\left(\begin{bmatrix} ate & b+e \\ c+g & d+h \end{bmatrix}\right) = (ate)x^3 + (b+e)x^2 + (c+g)x + d+h$$

$$T\left(\begin{bmatrix} ay \\ cd \end{bmatrix}\right) + T\left(\begin{bmatrix} ex \\ fg \end{bmatrix}\right) = (ax^3 + bx^2 + cx + d) + (ex^3 + fx^2 + gx + h)$$

$$T\left(k\begin{bmatrix} ay \\ cd \end{bmatrix}\right) = T\left(\begin{bmatrix} ka & kb \\ hc & kd \end{bmatrix}\right) = kax^3 + kbkx^2 + kcx + dk = k T\left(\begin{bmatrix} ay \\ cd \end{bmatrix}\right)$$

12. Find the eigenvalues and eigenvectors of (10)

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \quad (2I - A) = \det \begin{bmatrix} x-1 & -2 \\ 0 & x-2 \end{bmatrix} = 0$$

$\frac{1}{3}$

$$(x-1)(x-2) = 0$$

$x = 1, 2$

$$\lambda=1 \quad \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}$$

$\lambda=2$

$$x_2 = 0$$

$$\det \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x_1 = t$$

$$x_2 = t$$

$$t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$x_1 - 2t = 0$$

$$x_1 = 2t$$

(1,0)

$$t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

(3,0)

$$x_8 = t$$

$$x_7 = s$$

$$x_6 = w$$

$$x_5 + w + 2s + 3t = 0$$

$$x_5 = -w - 2s - 3t$$

$$3x_4 + 6(-w - 2s - 3t) + 7w + s = 0$$

$$3x_4 = -w + 4s + 18t$$

$$x_4 = -\frac{w}{4} + \frac{4}{4}s + 6t$$

$$2x_3 + 7\left(-\frac{w}{4} + \frac{4}{4}s + \frac{6}{4}t\right) + 2(-w - 2s - 3t) + 6w + t = 0$$

$$2x_3 + -\frac{7}{4}w + \frac{28}{4}s + 42t - 2w - 4s - 6t + 6w + t = 0$$

$$2x_3 - \frac{11}{4}w + \frac{61}{4}s + 37t = 0$$

$$2x_3 =$$

$$\left[ \begin{array}{ccccc} 2 & 4 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{array} \right] \rightarrow \left[ \begin{array}{ccccc} 2 & 4 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{array} \right] \rightarrow 4$$