

I. Give complete definitions for each:

a. A set of vectors  $v_1, v_2, \dots, v_n$  is *linearly independent*b. A function  $T$  from a vector space  $V$  to a vector space  $W$  is a *linear transformation*c. The matrices  $A$  and  $B$  are *similar*d. The scalar  $\lambda$  is an *eigenvalue* of the matrix  $A$ 

II. Problems: (Justify answers.)

1. Compute

$$\det \begin{bmatrix} 0 & 1 & 0 & 2 \\ 4 & -3 & 2 & 6 \\ -3 & 0 & 0 & 6 \\ 0 & 5 & 0 & -4 \end{bmatrix} = -2 \det \begin{bmatrix} 0 & 1 & 2 \\ -3 & 0 & 6 \\ 0 & 5 & -4 \end{bmatrix} = (-2)(3) \det \begin{bmatrix} 1 & 2 \\ 5 & -4 \end{bmatrix} = -6(-4 - 10) = 84$$
8

2. Find all solutions for the linear system:

$$x_1 + 2x_2 + 3x_3 = 0$$

$$-x_1 + -2x_2 + -2x_3 = 3$$

$$3x_1 + 6x_2 + 11x_3 = 6$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ -1 & -2 & -2 & 3 \\ 3 & 6 & 11 & 6 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 6 \end{array} \right]$$
B

$$\rightarrow \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_3 = 3 \quad x_2 = t$$

$$x_1 + 2t + 3 = 0$$

$$x_1 = -2t - 3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix}$$

$$-2t - 9$$

$$t$$

$$3$$

XO

3. Find the eigenvalues of

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \det \begin{bmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix} = (-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 1 \\ = \lambda^2 - 2\lambda + 2$$

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm \sqrt{-1} = 1 \pm i$$

4. The matrix A and its row-echelon form are:

$$\textcircled{20} \quad \begin{bmatrix} 2 & 1 & 2 & 0 & 3 \\ 4 & 4 & 5 & -2 & 7 \\ -6 & 1 & -4 & -2 & -5 \\ 6 & 11 & 10 & 2 & 23 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 2 & 0 & 3 \\ 0 & 2 & 1 & -2 & 1 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a. The rank of A is 3.

b. The row space is a 3 dimensional subspace of  $\mathbb{R}^5$ .

c. The column space (range) is a 3 dimensional subspace of  $\mathbb{R}^4$ .

d. The null space is a 2 dimensional subspace of  $\mathbb{R}^5$ .

e. Find a basis for the column space of A.

$$\begin{bmatrix} 2 \\ 4 \\ 8 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 11 \\ 1 \\ 10 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -2 \\ 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} 3 \\ -1/2 \\ 1/2 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ -4 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

f. Find a basis for the null space.

$$x_5 = t \quad x_1 = -t \quad x_3 = s$$

$$2x_2 + s - 2(-t) + t = 0$$

$$2x_2 = -s - 3t$$

$$x_2 = -\frac{s}{2} - \frac{3}{2}t$$

$$2x_1 + \left(-\frac{s}{2} - \frac{3}{2}t\right) + 2s + 3t = 0$$

$$2x_1 = -\frac{3}{2}s - \frac{3}{2}t$$

$$x_1 = -\frac{3}{4}s - \frac{3}{4}t$$

5. Find the inverse of

$$\begin{bmatrix} 2 & 4 & 5 \\ -6 & -10 & -10 \\ 2 & 0 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 5 & 1 & 0 & 0 \\ -6 & -10 & -10 & 0 & 1 & 0 \\ 2 & 0 & -4 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 5 & 1 & 0 & 0 \\ 0 & 2 & 5 & 3 & 1 & 0 \\ 0 & -4 & -9 & -1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 5 & 1 & 0 & 0 \\ 0 & 2 & 5 & 3 & 1 & 0 \\ 0 & 0 & 1 & 5 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 0 & -24 & -10 & -5 \\ 0 & 2 & 0 & -22 & -9 & -5 \\ 0 & 0 & 1 & 5 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 & 20 & 8 & 5 \\ 0 & 2 & 0 & -22 & -9 & -5 \\ 0 & 0 & 1 & 5 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 10 & 4 & 5/\sqrt{2} \\ -11 & -9/\sqrt{2} & 5/\sqrt{2} \\ 5 & 2 & 1 \end{bmatrix}$$

For 3  
completely  
done

6. Find the eigenvalues and eigenvectors of

$$\begin{bmatrix} 2 & 0 \\ 3 & 4 \end{bmatrix} \quad 2 \quad 4$$

$$0 \ 0$$

$$3 \ 2$$

$$x_2 = t$$

$$x_1 = -\frac{2}{3}t$$

$$\textcircled{2} \quad \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 \\ 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$x_2 = t$$

$$x_1 = 0$$

$$\textcircled{3} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Same  
Same  
Same  
vert.

7. Find a basis for the subspace spanned by the vectors  $(1, 2, 3, -2)$ ,  $(2, 4, 7, -2)$ , and  $(0, 0, 1, 2)$ .

This is a 2 dimensional subspace of  $\mathbb{R}^4$   $\leftarrow$  may say  $\mathbb{R}^3$

$$\begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & 4 & 7 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(1, 2, 3, -2), (0, 0, 1, 2) \text{ at } \mathbb{R}^3$$

8. Consider the set of polynomials  $p(x) = a_0 + a_1x + a_2x^2$  for which  $a_1 = 0$ .

a. Give an example of such a polynomial.

b. Show that this set is a subspace of  $P_2$ .

c. Find a basis and the dimension.

$$a. \quad 1 + x^2$$

$$b. \quad (a_0 + a_2x^2) + (b_0 + b_2x^2) = (a_0 + b_0) + (a_2 + b_2)x^2$$

$$r(a_0 + a_2x^2) = r a_0 + r a_2 x^2$$

$$c. \quad 1, x^2, 2$$

last

subset