

Name _____

Key

[S-U ? ___]

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(perfect paper)

I. Give complete definitions for each

a. The set of vectors v_1, v_2, \dots, v_n is linearly independent

b. The null space of the matrix A is

c. The vectors b_1, b_2, \dots, b_n form a basis for a vector space V if and only if

d. The dimension of a vector space V is

e. The scalar λ is an eigenvalue of the matrix A if and only if

f. The rank of a matrix A is

II. Prove TWO (AND fill-in where necessary):

- a. $\text{Nul}(A) = \text{Nul}(A^T A)$.
- b. The span of (or set spanned by) the vectors v_1, v_2, \dots, v_n in a vector space V is a subspace of V .
- c. If b_1, b_2, \dots, b_n is a basis for a vector space V , then every vector v in V can be written as a linear combination of b_1, b_2, \dots, b_n in a unique way.
- d. The null space of the matrix A is a subspace of \mathbb{R}^n .

III. Problems (justify answers)

1. $A = \begin{bmatrix} 1 & 2 & -3 & 0 & 1 \\ -2 & -3 & 8 & -1 & -1 \\ 4 & 9 & -10 & 0 & 7 \\ -5 & -12 & 11 & 3 & -5 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & 2 & -3 & 0 & 1 \\ 0 & 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

a. Find a basis for the row space of A:

$$(1, 2, -3, 0, 1) \quad (0, 1, 2, -1, 1) \quad (0, 9, 1, 2)$$

b. Find a basis for the column space of A:

$$(1, -3, 4, 5) \quad (2, -3, 9, -12) \quad (0, -1, 9, 3)$$

c. The row space of A is a 3 dimensional subspace of \mathbb{R}^5 .

d. The column space of A is a 3 dimensional subspace of \mathbb{R}^4 .

e. The rank of A is 3 ~~4~~ several missed!

f. The nullspace of A is a 2 dimensional subspace of \mathbb{R}^5 .

g. Find a basis for the null space of A:

$$x_3 = t$$

$$x_4 = -2t$$

$$x_5 = s$$

$$x_2 + 2s - (-2s) + t = 0$$

$$x_2 = -2s + 3t$$

$$x_1 + 2(-2s + 3t) - 3s + t = 0$$

$$x_1 + 4s - 6t - 3s + t = 0$$

$$x_1 = 2s + 5t$$

$$s \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ -3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

(4)

2. find the inverse of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 7 & 12 \\ -2 & -2 & -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 7 & 12 & 0 & 1 & 0 \\ -2 & -2 & -1 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & -3 & 1 & 0 \\ 0 & 2 & 5 & 2 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & -3 & 1 & 0 \\ 0 & 0 & -1 & 8 & -2 & 1 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 25 & -6 & 3 \\ 0 & 1 & 0 & 21 & -5 & 3 \\ 0 & 0 & -1 & 8 & -2 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -17 & 4 & -3 \\ 0 & 1 & 0 & 21 & -5 & 3 \\ 0 & 0 & -1 & 8 & -2 & 1 \end{array} \right]$$

adj (cont'd)

$$\begin{bmatrix} -17 & 4 & -3 \\ 21 & -5 & 3 \\ -8 & 2 & -1 \end{bmatrix}$$

3. The rank of the following matrix is:

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 1 \\ 4 & 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -3 \\ 0 & 3 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -3 \\ 0 & 0 & -2 \end{bmatrix}$$

(3)

5

all but 1

4. Find a basis for the space spanned by the vectors $(1, 2, 0, 3, 4)$, $(-4, -8, 1, -10, -18)$, and $(2, 4, -1, 4, 10)$. This is a (2) dimensional subspace of \mathbb{R}^5 .

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 4 \\ -4 & -8 & 1 & -10 & -18 \\ 2 & 4 & -1 & 4 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & -1 & -2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(8)

$$(1, 2, 0, 3, 4) \ (0, 0, 1, 2, -2)$$

5. Find all solutions for the system of equations $x_1 + 3x_2 - x_4 = 9$, $x_1 + 3x_3 + x_4 = 3$.

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 1 & 0 & 3 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & -3 & 3 & 2 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 1 & -1 & -\frac{2}{3} \end{array} \right]$$

all but 2

$$x_4 = t \quad x_3 = s$$

$$-3x_2 + 3s + 2t = -6$$

$$-3x_2 = -6 - 2t - 3s$$

$$x_2 = 2 + \frac{2}{3}t + s$$

$$x_1 + 3(2 + \frac{2}{3}t + s) - t = 9$$

$$x_1 + 6 + 2t + 3s - t = 9$$

$$x_1 = 3 - t - 3s$$

$$\begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2/3 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

8

8

6. Find the eigenvalues of

$$\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$$

$$\det \begin{bmatrix} 1-\lambda & 2 \\ -2 & -4-\lambda \end{bmatrix} = (1-\lambda)(-4-\lambda) + 4 = -4 + 3\lambda + \lambda^2 + 4 = \lambda^2 + 3\lambda = 0$$

$$\lambda(\lambda+3)=0$$

$$\lambda=0, -3$$

all but 1

8

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7. Is $(1, 4, -6)$ in the span of $(1, 2, 0)$, $(2, 5, -4)$ and $(-3, -5, -4)$? 8

$$a(1, 2, 0) + b(2, 5, -4) + c(-3, -5, -4) = (1, 4, -6)$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 2 & 5 & -5 & 4 \\ 0 & -4 & -4 & -6 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & -4 & -4 & -6 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -2 \end{array} \right]$$

No!

all but 6

8. Find the eigenvalues and eigenvectors for the matrix 10

$$\left[\begin{array}{ccc} 1 & 2 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{array} \right] \lambda = 1, 3, 2$$

$$\left[\begin{array}{ccc} 1-\lambda & 2 & 3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & 2-\lambda \end{array} \right] \lambda = 1$$

$$\left[\begin{array}{ccc} -2 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{array} \right] \rightarrow \left[\begin{array}{ccc} 0 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right]$$

$$\lambda = 2$$

$$\left[\begin{array}{ccc} -1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right]$$

$$\begin{aligned} x_3 &= t \\ x_2 &= -2t \\ -x_1 + 2(-2t) + 3t &= 0 \\ -x_1 - 2t &= 0 \\ x_1 &= -2t \quad x_1 = -1 \end{aligned}$$

$$\begin{aligned} x_3 &= 0 \\ x_2 &= 0 \\ x_1 &= t \end{aligned}$$

$$t(1, 0, 0)$$

all

$$\begin{aligned} x_3 &= 0 \\ x_2 &\neq t \\ -2x_1 + 2t &= 0 \end{aligned}$$

$$t(4, 0, 0)$$

all

9. Use matrix methods to find the equation of the least squares straight line through the points $(-1, 1.2)$, $(0, 2.5)$, $(1, 3.1)$ 10

$$y = \beta_0 + \beta_1 x$$

$$X = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \quad Y = \begin{bmatrix} 1.2 \\ 2.5 \\ 3.1 \end{bmatrix}$$

1 all
4 close

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.2 \\ 2.5 \\ 3.1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 6.8 \\ 1.9 \end{bmatrix}$$

$$y = 2.26 + .95x$$

$$\beta_1 = \frac{1.9}{2} = .95, .95$$

$$\beta_0 = \frac{6.8}{3} = 2.26$$

10. The vectors $(1, 2)$ and $(2, 1)$ form a basis for \mathbb{R}^2 .

a. Justify this statement.

b. Write the coordinates of the vector $(-1, 4)$ relative to this basis.

8 12, b. i. and

b. $\begin{pmatrix} -1 \\ 4 \end{pmatrix} = a \begin{pmatrix} 1 \\ 2 \end{pmatrix} + b \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\left[\begin{array}{cc|c} 1 & 2 & -1 \\ 2 & 1 & 4 \end{array} \right] \xrightarrow{\text{Row operations}} \left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & -3 & 6 \end{array} \right]$$

$$B = -2$$

$$a - 4 = -1 \quad a = 3$$

(3, -2)

↙ must
used

8
full

11. Find the equation of the curve of the form $y = a + bx + c e^x$ which passes through the points $(0, 2)$, $(1, 3)$ and $(3, 5)$. (Set up only! Do not carry out calculations.)

$$X = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & e \\ 1 & 3 & e^3 \end{bmatrix} \quad Y = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

11 all

most close

12. Let P_4 be the set of all polynomials of degree 4 or less.

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a. Give a basis for this space. What is the dimension? \leftarrow 10 all

b. Are $1 - x + x^4$, $1 + x + x^3$, and $1 + x$ linearly independent? (4) must used to answer

c. Let D be the transformation $D(p) = \text{derivative of } p$.

i. What is $D(3x^2 + 3x - 2)$? \leftarrow ② all

use it ii. Explain why this is a linear transformation from $\underline{\quad}$ to $\underline{\quad}$. \leftarrow a couple close.

iii. Is this transformation one-to-one? Justify.

a. $1, x, x^2, x^3, x^4, 5$

b. $a(1-x) + b(x)$

$$a(1-x+x^4) + b(1+x+x^3) + c(1+x)^2 = 0$$

$$a+b+c=0$$

$$ab=0$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right] \xrightarrow{\text{Row operations}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right]$$

$$-a+b+c=0$$

$$c=0 \quad y_3$$

$$a \quad = 0$$

$$b \quad = 0$$

$$0 \quad b \quad = 0$$

yes

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right]$$

c. $D(3x^2 + 3x - 2) = 6x + 3$

(4) $P_4 \rightarrow P_3$ or P_3 dof 5 = 5 odd, eval.

(6) no $D(x^4) = D(x)$

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