

I. Give complete definitions for each of the following: See

a. linear independence The vectors v_1, \dots, v_n are linearly independent iff $a_1 v_1 + \dots + a_n v_n = 0$ only if $a_1 = 0, \dots, a_n = 0$

b. A function from A into B assigns a unique element of B to every element of A.

c. Basis for a vector space B_1, \dots, B_n is a basis for a vector space if they are linearly independent and span it.

d. linear transformation $L: A \rightarrow B$ is a linear transformation iff $L(a+cb) = L(a) + cL(b) \forall a, b \in A, c \in \mathbb{R}$ and $L(va) = vL(a) \forall v \in \mathbb{R}, a \in A$.

e. null space of a linear transformation
 The null space of T is defined by
 $N_T = \{ a \in A \mid T(a) = 0 \}$

II. Give reasons in each part: See

a. Find the orthogonal projection of (2,3,1) onto (1,1,2)



$$\|A\| \cos \theta \cdot \frac{B}{\|B\|} = \frac{A \cdot B}{B \cdot B} B = \frac{2+3+2}{1+1+4} (1,1,2) = \frac{7}{6} (1,1,2)$$

b. Are (2,3,4) and (-3,1,1) orthogonal? $= (\frac{7}{6}, \frac{7}{6}, \frac{14}{6})$

$$(2,3,4) \cdot (-3,1,1) = -6 + 3 + 4 = 1 \quad \text{No.}$$

c. What is the angle between the planes (or its cosine)

$$2x + y + 2z = 1 \quad \text{and} \quad x + y + 2z = 3.$$

$$N_1 = (2, 1, 2) \quad N_2 = (1, 1, 2) \quad \cos \theta = \frac{7}{3\sqrt{6}}$$

$$\|N_1\| = \sqrt{4+1+4} = 3 \quad N_1 \cdot N_2 = 2+1+4 = 7$$

$$\|N_2\| = \sqrt{1+1+4} = \sqrt{6} = 2$$

d. $\begin{vmatrix} 2 & 6 \\ 3 & -2 \end{vmatrix} = -4 - 18 = -22$

e. $\begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix}^{-1} = \frac{1}{-15} \begin{bmatrix} 0 & -5 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 \\ 1/5 & -2/15 \end{bmatrix}$

f. Fill in with an elementary matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 2 \\ 0 & -1 \end{bmatrix}$$

g. Is $T(x,y) = (x+2y, x-3)$ a linear transformation? Justify.

NO $T(x_1, y_1) + T(x_2, y_2) = (x_1 + 2y_1, x_1 - 3) + (x_2 + 2y_2, x_2 - 3)$
 $= (x_1 + x_2 + 2(y_1 + y_2), (x_1 + x_2) - 6)$

h. If $T(1,0) = (2,4)$ and $T(0,1) = (6,-1)$ then $T(x,y) = (2x+6y, 4x-y)$.

$$T(x,y) = \begin{bmatrix} 2 & 6 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+6y \\ 4x-y \end{bmatrix}$$

i. Is $T(x,y,z) = (x-y, y, 2y)$ 1-1 (nonsingular)?

$N_T = \{ (0,0,z) \mid z \in \mathbb{R} \}$ $\dim N_T = 1$ $\begin{matrix} z \neq 0 \\ -3 \end{matrix}$
 NO!

$y=0, x=0, z \neq 0$

j. Find the rank of $T(x,y,z) = (x+y, x-2y)$.

$x+y=0$ $x+2x=0$ $\text{rank} = 2$
 $x-2y=0$ $x=0$ $y=0$

k. If $T(x,y) = (2x+y, x+y)$, find T^{-1} .

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \quad T^{-1}(x,y) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-y \\ -x+2y \end{bmatrix}$$

l. What is the matrix associated with $T(x,y,z) = (2x, 3x-z)$ with the natural basis in both the domain and range.

$T(1,0,0) = (2, 3)$
 $T(0,1,0) = (0, 0)$
 $T(0,0,1) = (0, -1)$ $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 0 & -1 \end{bmatrix}$

2. Show: $\dim \mathbb{R}^n = n$.

E_1, \dots, E_n span \mathbb{R}^n
and are linearly independent so a basis.
so $\dim = n$

2. Write $(-9, -1)$ as a linear combination of $(1, 1)$ and $(6, 2)$.

$$\begin{aligned} (-9, -1) &= a(1, 1) + b(6, 2) \\ &= (a + 6b, a + 2b) \\ a + 6b &= -9 \\ a + 2b &= -1 \end{aligned}$$

$$\begin{aligned} 4b &= -8 & b &= -2 \\ a &= -1 - 2(-2) & &= 3 \end{aligned}$$

3. Are $(1, 1, 2)$, $(1, -1, 0)$, and $(0, 1, 2)$ linearly independent?

$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 1 \begin{vmatrix} -1 & 0 \\ 1 & 2 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} + 0 = -2 \neq 0 \text{ yes}$$

4. Is $(4, 1, 9)$ in the subspace generated by $(2, 1, 1)$ and $(1, 2, 6)$?

$$(4, 1, 9) = a(2, 1, 1) + b(1, 2, 6)$$

$$\begin{aligned} 4 &= 2a + b & 4 &= 2(-6) + 2 & \text{no} \\ 1 &= a + 2b & 4b &= 8 & a = 1 - 2b \\ 9 &= a + 6b & b &= 2 & 2 - 4 = -2 \end{aligned}$$

5. The transpose of the matrix $\begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 4 & -2 \end{bmatrix}$ is $\begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & -2 \end{bmatrix}$

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$$\begin{aligned} \text{a. } \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 1 & 1 & 2 \end{vmatrix} &= 1 \begin{vmatrix} 1 & 0 & 1 \\ 2 & 3 & 4 \\ 1 & 1 & 2 \end{vmatrix} + 0 + 1 \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 1 & 2 \end{vmatrix} + 0 \\ &= 1 \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} + 0 + 1 \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \\ &= 2 - 1 + 6 + 3 \\ &= 10 \end{aligned}$$

II. b. Find the vector equation of the points on the line between $(2,3,4)$ and the plane $x + y + 2z = 1$ ^{perpendicular}

$$N = (1, 1, 2)$$

$$X = (2, 3, 4) + t(1, 1, 2) = (2+t, 3+t, 4+2t)$$

$$X = (2, 3, 4) + t(1, 1, 2) \quad -2 \leq t \leq 0$$

$$2+t + 3+t + 2(4+2t) = 1$$

$$13 + t(6) = 1$$

$$6t = -12 \quad t = -2$$

$$X = (2, 3, 4) + t(-2, -2, -4) \quad 0 < t < 1$$

c. Find the inverse of $\begin{bmatrix} -1 & 0 & 2 \\ 2 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$.

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$$\det = 2 \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} = 2(-1) = -2$$

$$\text{adj} = \begin{bmatrix} \begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & 2 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} -1 & 2 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} +2 & -4 & 0 \\ 0 & -2 & 0 \\ -2 & 7 & -1 \end{bmatrix}$$

$$\text{inverse} = -\frac{1}{2} \begin{bmatrix} 2 & 0 & -2 \\ -4 & -2 & 7 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & -7/2 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 2 \\ 2 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3/2 & 0 & 1 \\ 2 & 1 & -7/2 \\ 0 & 0 & 1 \end{bmatrix}$$

d. What is the dimension of the subspace of \mathbb{R}^3 generated by $(1,0,1)$, $(2,1,1)$, $(3,1,2)$, and $(-6,-2,-4)$?

$$(1,0,1) \quad (3,1,1) \quad \text{and}$$

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 1 \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + 0 + 1 \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = 1 - 1 = 0 \quad \text{dep.}$$

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -6 & -2 & -4 \end{vmatrix} = 0$$

dim 2

II. cont.

- e. Find the solution(s) of
- $$\begin{aligned} x + y + z &= 2 \\ x + 2y + 2z &= 3 \\ y + z &= 1 \end{aligned}$$

and describe the solution space geometrically.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & 1 \end{array} \right] \xrightarrow{z=0} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\xrightarrow{\textcircled{1}-\textcircled{2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{aligned} z &= t \\ x &= 1 \\ y + z &= 1 \\ y &= 1 - t \end{aligned}$$

$$(x, y, z) = (1, 1-t, t) = (1, 1, 0) + t(0, -1, 1)$$

line thru (1, 1, 0) in (0, -1, 1) direction.

- f. Prove: N_T , the null space of a linear transformation T from A into B , is a subspace of A .

$$T(u_1) = 0 \quad T(u_2) = 0$$

$$T(u_1 + u_2) = T(u_1) + T(u_2) = 0 + 0 = 0 \Rightarrow u_1 + u_2 \in N_T$$

$$T(u) = 0$$

$$T(rv) = rT(u) = 0 \quad \forall r \Rightarrow rv \in N_T$$

- g. Find (v_2, i_2) as a function of (v_1, i_1) .

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/5 & 1 \end{bmatrix} \begin{bmatrix} 1 & -10 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ u_1 \end{bmatrix} = \begin{bmatrix} 7/5 & -16 \\ -1/5 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ u_1 \end{bmatrix}$$

$$\begin{aligned} -3 \begin{bmatrix} 1 & -10 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/5 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ u_1 \end{bmatrix} &= \begin{bmatrix} 1 & -10 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1/5 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ x_1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -12 \\ -1/5 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ i_1 \end{bmatrix} \end{aligned}$$

II. cont.

h. The matrix $\begin{bmatrix} 2 & 3 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ is associated with a linear transformation from \mathbb{R}^3 into \mathbb{R}^2 with respect to the natural basis in \mathbb{R}^3 and the basis $B_1 = (2, 3)$ and $B_2 = (1, -1)$ in \mathbb{R}^2 . Find $T(x, y, z)$ explicitly. i.e. $T(x, y, z) = (x+7y+10z, 9x+8y+10z)$.

$$\begin{aligned} T(1, 0, 0) &= 2(2, 3) - 3(1, -1) \\ &= (4-3, 6+3) = (1, 9) \end{aligned}$$

$$\begin{aligned} T(0, 1, 0) &= 3(2, 3) + (1, -1) \\ &= (6+1, 9-1) = (7, 8) \end{aligned}$$

$$\begin{aligned} T(0, 0, 1) &= 4(2, 3) + 2(1, -1) = \\ &= (8+2, 12-2) = (10, 10) \end{aligned}$$

$$T(x, y, z) = \begin{bmatrix} 1 & 7 & 10 \\ 9 & 8 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+7y+10z \\ 9x+8y+10z \end{bmatrix}$$

i. The set M_3 of 3×3 matrices form a vector space.

a. Verify that the set of upper triangular matrices

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

form a subspace.

b. Find a basis for this subspace.

$$a. \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} + \begin{bmatrix} g & h & i \\ 0 & j & k \\ 0 & 0 & l \end{bmatrix} = \begin{bmatrix} a+g & b+h & c+i \\ 0 & d+j & e+k \\ 0 & 0 & f+l \end{bmatrix} \in T$$

$$r \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} = \begin{bmatrix} ra & rb & rc \\ 0 & rd & re \\ 0 & 0 & rf \end{bmatrix} \checkmark$$

$$b. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$