

MATH 332
TEST II
NOVEMBER 17, 1981

NAME _____

KEY

44
33
12

Show complete solutions with justifications.

OMIT 1
(up to 10 pts)
First left after 50
Several rows
done, but
were checking
qed 83

(10) 1. Carefully state the following definitions:

a. the vectors $\underline{v}_1, \dots, \underline{v}_n$ are linearly independent iff

$$a_1 \underline{v}_1 + \dots + a_n \underline{v}_n = \underline{0}$$

has only $a_1 = 0, \dots, a_n = 0$ as soln.

b. the subset S of a vector space V is a subspace of V iff

it is a vector space itself with the same operations as V .

(6) 2. Is $(8, 9, 6)$ in the ^{span} of the vectors $(1, 3, 2)$ and $(2, 1, 1)$?

6 0 0
13.1
12.4
10.77

$$(8, 9, 6) = a(1, 3, 2) + b(2, 1, 1)$$

$$\begin{aligned} a + 2b &= 8 \\ 3a + b &= 9 \\ 2a + b &= 6 \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 8 \\ 3 & 1 & 9 \\ 2 & 1 & 6 \end{array} \right]$$

$$\begin{aligned} \textcircled{2} - 3\textcircled{1} & \left[\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & -5 & -15 \\ 0 & -3 & -10 \end{array} \right] \\ \textcircled{3} - 2\textcircled{1} & \left[\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & -3 & -10 \end{array} \right] \end{aligned}$$

no.

$$\left[\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & 1 & 3 \\ 0 & 3 & 10 \end{array} \right]$$

$$\textcircled{3} - 3\textcircled{2} \left[\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & 1 & 3 \\ 0 & 1 & 10 \end{array} \right]$$

34

(18) 3. Let $\underline{v} = (1, 2, 1, 1)$, $\underline{w} = (0, 1, 1, 3)$, and $\underline{u} = (3, -1, -1, 1)$. Find:

a. $3\underline{v} + \underline{u} = (3, 6, 3, 3) + (3, -1, -1, 1) = (6, 5, 2, 4)$

b. $\underline{v} \cdot \underline{u} = 3 + 2 - 1 + 1 = 1$

c. $\|\underline{v}\| = \sqrt{1+4+1+1} = \sqrt{7}$

d. Euclidean distance between \underline{w} and \underline{u} .

$$\underline{w} - \underline{u} = (-3, 2, 2, 2)$$

$$\|\underline{w} - \underline{u}\| = \sqrt{3^2 + 2^2 + 2^2 + 2^2} = \sqrt{23}$$

e. $\frac{1}{\|\underline{u}\|} \underline{u} = \left(\frac{3}{\sqrt{12}}, -\frac{1}{\sqrt{12}}, -\frac{1}{\sqrt{12}}, \frac{1}{\sqrt{12}} \right)$

$$\|\underline{u}\| = \sqrt{9+1+1+1} = \sqrt{12}$$

35

f. Are \underline{w} and \underline{u} orthogonal? Justify.

$$\underline{w} \cdot \underline{u} = 0 - 1 - 1 + 3 = 1 \neq 0$$

NO!

36

(5) 4. Find the orthogonal projection of $(1, 2, 0)$ onto $(2, 0, 2)$.

$$\frac{\underline{u} \cdot \underline{v}}{\underline{v} \cdot \underline{v}} \underline{v} = \frac{2}{\sqrt{8}} (2, 0, 2) = \left(\frac{2}{\sqrt{8}}, 0, \frac{2}{\sqrt{8}} \right)$$

$$\frac{1}{2} (2, 0, 2)$$

$$\left(\frac{1}{2}, 0, \frac{1}{2} \right)$$

—/

- (6) 5. Find the dimension of the solution space of the following system of equations and give a basis.

$$x_1 + x_2 + x_3 + x_4 = 0$$

$$2x_1 + x_2 - x_3 = 0$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{\text{②} - 2\text{①}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -3 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 2 \end{bmatrix}$$

$$x_4 = s$$

$$x_2 = -3t - 2s$$

$$x_3 = t$$

$$x_1 = 3t + 2s - t - s$$

$$= 2t + s$$

$$(x_1, x_2, x_3, x_4) = t(2, -3, 1, 0) + s(1, -2, 0, 1) \quad \text{②}$$

- (10) 6. Consider the set of all 2×2 matrices of the form $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$.

a. Show that this is a subspace of M_{22} .

b. What is the dimension of this subspace? Justify.

$$\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} + \begin{bmatrix} d & 0 \\ e & f \end{bmatrix} = \begin{bmatrix} a+d & 0 \\ b+e & c+f \end{bmatrix}$$

$$k \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} = \begin{bmatrix} ka & 0 \\ kb & kc \end{bmatrix}$$

Basis $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \underline{3}$

- (6) 7. Do the vectors $(1,0,3)$, $(3,6,1)$, and $(2,2,1)$ form a basis for \mathbb{R}^3 ? Show work.

$$\begin{array}{c|ccc|c} 1 & 0 & 3 & 1 & 0 \\ 3 & 6 & 1 & 3 & 6 = 6 + 0 + 18 - 36 - 2 = 0 \\ 2 & 2 & 1 & 2 & 2 \end{array} \quad \ominus$$

$$= 24 - 38 \neq 0 \text{ yes.}$$

$$= -14$$

- (6) 8. Find a basis for the column space of the matrix

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 4 & 3 & 1 \\ 1 & 3 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (1, 3, 1, 1) \quad (0, 1, 1, 0)$$

- (6) 9. Find a basis for the subspace of \mathbb{R}^4 spanned by the vectors $(4,5,1,2)$, $(1,2,0,1)$, and $(2,1,1,0)$. What is the dimension of this subspace?

$$\begin{bmatrix} 4 & 5 & 1 & 2 \\ 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 4 & 5 & 1 & 2 \\ 2 & 1 & 1 & 0 \end{bmatrix}$$

$$\begin{array}{l} \textcircled{2} - 4\textcircled{1} \\ \textcircled{3} - 2\textcircled{1} \end{array} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -3 & 1 & -2 \\ 0 & -3 & 1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(4, 3, 0, 1) \quad (0, -3, 1, -2) \quad \textcircled{2}$$

- (6) 10. Justify this statement: The vectors (a,b,c) , (d,e,f) , and (g,h,i) lie in the same plane (through $(0,0,0)$) if $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 0$.

if $\det = 0$
 rows of mat are dep.
 Hence span 2 dim space = plane.

- (10) 11. Are the following sets of vectors in P_2 linearly independent? Justify. Do they form a basis?

- a. $x^2, 1-x^2, x+2x+x^2, 2-3x$. too many.
 b. $1+x+x^2, 1+x, x-x^2$.

$$a(1+x+x^2) + b(1+x) + c(x-x^2) = 0$$

$$a+b+0 = 0$$

$$a+b+c = 0$$

$$a+0-c = 0$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{matrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{matrix} = 1$$

$$\det = 0 + 0 - 1 - (-1 + 1 + 0) = -1 \neq 0$$

only kind sol yes

- (5) 12. Show that in a vector space, the negative of a vector is unique. (Use only the axioms, not later theorems.)

$$v+w=0$$

$$v+u=0$$

$$u+v+w=0$$

$$v+u+w=0$$

$$0+w=0$$

$$w=u$$

(5) 13. Suppose b_1, \dots, b_n is a basis for a vector space V . Show that any vector in V can be written as a unique linear combination of b_1, \dots, b_n .

$$a_1 b_1 + \dots + a_n b_n = v$$

$$c_1 b_1 + \dots + c_n b_n$$

$$a_1 b_1 + \dots + a_n b_n - c_1 b_1 - \dots - c_n b_n = 0$$

$$(a_1 - c_1) b_1 + \dots + (a_n - c_n) b_n = 0$$

b are indep

$$\Rightarrow a_i - c_i = 0$$

$$a_i = c_i$$

$$a_1 = c_1$$

i

$$a_n = c_n$$