

## Part II

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125 pts

know work. Use complete sentences in definitions. You may assume that the standard examples -  $\mathbb{R}^n$ ,  $P_n$ ,  $M_{m,n}$  - are vector spaces.

1. Give complete definitions for each of the following (25 points)

1. The vectors  $v_1, \dots, v_n$  are linearly independent

2. A subset of a vector space is a subspace

3. The span of the vectors  $v_1, \dots, v_n$

4. A basis for a vector space

5. The rank of a matrix

II. Give proofs for each of the following (complete the statement where there are blanks): (40 points)

1. For any vectors  $\vec{u}, \vec{v}, \vec{w}$  in  $\mathbb{R}^n$ ,  $(\vec{u} + \vec{v}) \cdot \vec{w} = (\vec{u} \cdot \vec{w}) + (\vec{v} \cdot \vec{w})$

$$((u_1, \dots, u_n) + (v_1, \dots, v_n)) \cdot (w_1, \dots, w_n) = (u_1 + v_1, \dots, u_n + v_n) \cdot (w_1, \dots, w_n)$$

$$= (u_1 + v_1)w_1 + \dots + (u_n + v_n)w_n$$

$$(u_1, \dots, u_n) \cdot (w_1, \dots, w_n) + (v_1, \dots, v_n) \cdot (w_1, \dots, w_n)$$

$$= u_1w_1 + \dots + u_nw_n + v_1w_1 + \dots + v_nw_n$$

$$= (u_1 + v_1)w_1 + \dots + (u_n + v_n)w_n$$

2. The solution set of  $A\vec{x} = \vec{0}$  is a subspace of  $\mathbb{R}^n$ , where  $A$  is  $n \times m$ .

$$A(x_1 + x_2) = Ax_1 + Ax_2 = 0 + 0 = 0$$

$$A(kx) = k(Ax) = k0 = 0$$

3. If a set of  $\begin{matrix} n \\ \text{non zero} \end{matrix}$  vectors is linearly dependent, then at least one can be written as a linear combination of the others.

L.D  $\Rightarrow \exists c_1, \dots, c_n \text{ not all } 0 \text{ s.t. } c_1\vec{v}_1 + \dots + c_n\vec{v}_n = 0$

Wlog assume  $c_1 \neq 0$ . so

$$\vec{v}_1 = -\frac{c_2}{c_1}\vec{v}_2 + \dots + -\frac{c_n}{c_1}\vec{v}_n$$

4. If  $\vec{v}_1, \dots, \vec{v}_n$  form a basis for a vector space  $V$ , then every vector in  $V$

can be expressed as a linear combination of  $\vec{v}_1, \dots, \vec{v}_n$  in only one

way. Suppose  $c_1\vec{v}_1 + \dots + c_n\vec{v}_n = d_1\vec{v}_1 + \dots + d_n\vec{v}_n$

$$(c_1 - d_1)\vec{v}_1 + \dots + (c_n - d_n)\vec{v}_n = 0$$

$$\vec{v}_1, \dots, \vec{v}_n \text{ li. } \Rightarrow c_1 - d_1 = 0, \dots, c_n - d_n = 0$$

$$\text{or } c_1 = d_1, \dots, c_n = d_n$$

## III. Problems: Justify answers.

(60 points)

1. Are the vectors  $(1, 2, 1)$ ,  $(2, 5, 3)$ , and  $(-1, 1, 8)$  linearly independent?

$$\begin{aligned} \det \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ -1 & 1 & 8 \end{bmatrix} &= 1 \begin{vmatrix} 5 & 3 \\ 1 & 8 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 1 & 8 \end{vmatrix} + (-1) \begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix} \\ &= 40 - 5 - 2(16 - 1) - (10 - 5) \\ &= 35 - 30 - 5 = 0 \quad \text{No!} \end{aligned}$$

2. Give a basis for  $P_4$ . What is the dimension of  $P_4$ ?

$$1, x, x^2, x^3, x^4$$

5

3. Consider the set of polynomials of the form  $a_1x + a_2x^2$ . Is this a subspace of  $P_2$ ?

$$\begin{aligned} \text{Yes } a_1x + a_2x^2 \\ + b_1x + b_2x^2 \\ \hline (a_1+b_1)x + (a_2+b_2)x^2 \\ k(a_1x + a_2x^2) = (ka_1)x + (ka_2)x^2 \end{aligned}$$

4. Find a basis for the space spanned by the vectors  $(1, 2, 0, 1)$ ,  $(3, 7, 3, 3)$ ,

$$(-2, -8, -12, -2)$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 7 & 3 & 3 \\ -2 & -8 & -12 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & -4 & -12 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(1, 2, 0, 1), (0, 1, 3, 0), \cancel{(3, 7, 3, 3)}$$

$$\frac{28}{4} \frac{4}{4} \frac{4}{4}$$

1.046

$$\frac{3}{2} \frac{1}{2} \frac{1}{2}$$

5. Find a basis and give the dimension for the solution space of the

systems

$$2x_1 + x_2 - x_3 + 4x_4 = 0$$

$$x_2 + 2x_3 - x_4 = 0.$$

$$\begin{bmatrix} 2 & 1 & -1 & 4 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$

$$x_4 = t$$

$$x_3 = s$$

$$x_2 + 2s - t = 0 \quad x_2 = t - 2s$$

$$2x_1 + (t - 2s) - s + 4t = 0$$

$$2x_1 + 5t - 3s = 0 \quad x_1 = \frac{3}{2}s - \frac{5}{2}t$$

6. Consider the set of matrices of the form  $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ .

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

a. Show that these form a vector space.

b. Find a basis for this space. What is the dimension?

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} + \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} a+d & b+e \\ 0 & c+f \end{bmatrix}$$

$$k \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} ka & kb \\ 0 & kc \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

7. Use the Gram-Schmidt process to find an orthonormal basis for the

space spanned by the vectors  $(1, 1, 0)$  and  $(3, 1, 2)$ .

$$v_1 = (1, 1, 0)$$

$$v_2 = (3, 1, 2) - \frac{(3, 1, 2) \cdot (1, 1, 0)}{(1, 1, 0) \cdot (1, 1, 0)} (1, 1, 0)$$

$$= (3, 1, 2) - \frac{4}{2} (1, 1, 0) = (3, 1, 2) - 2(1, 1, 0) = (1, -1, 2)$$

$$(1, 1, 0) (1, -1, 2)$$

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$$